Positivity in Hodge theory with applications to algebraic geometry[1]

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[1] Informal notes for the talks. A more complete set of notes together with references are in the mathematics web sites [G] and [GG].

- A polarized Hodge structure $(V, F^\bullet, Q)$ has two Hodge-Riemann bilinear relations

$$\begin{cases} \text{(HRI)} & Q(F^p, F^{n-p+1}) = 0 \\ \text{(HRII)} & Q(F^p, C\overline{F}^p) > 0 \end{cases}$$

  where $C =$ Weil operator $= i^{p-q}\,\mathrm{Id}$ on $V^{p,q} = F^p \cap \overline{F}^q$. Both are usually assumed but rarely used directly in general cohomological arguments.
- These give metrics in the Hodge bundles and the resulting curvatures have remarkable properties. Purpose of these talks is to discuss their curvatures and give applications to algebraic geometry.

- Anticipating what comes later the curvature matrices have the form $\Theta = A \wedge {}^t\overline{A}$ where A is the matrix of a holomorphic bundle mapping whose entries are holomorphic 1-forms. Thus Θ is a first order invariant and $\Theta = 0$ is a complex analytic condition.
- **Example:** Curvature of $\mathcal{O}_{\mathbb{P}^n}(1)$ is $\frac{(dz,dz)(z,z)-(dz,z)(z,dz))}{(z,z)^2}$.

———<∘>———

- Y = smooth quasi-projective variety; then $Y = \overline{Y} \backslash Z$ where $Z = \cup Z_i$ is a normal crossing divisor. Typical interesting properties that Y might have are:

(A) Y is of log general type; i.e., $K_{\overline{Y}}(\log Z)$ is big (independent of Z);

(B) $\Omega^1_{\overline{Y}}(\log Z)$ is big (also independent of Z);

(C) Y is hyperbolic (any non-constant holomorphic mapping $f : \Delta(r) \to Y$ has $r \leqq r_0(f'(0)) < \infty$).

Also

(C′) Y is algebraically hyperbolic: Smooth algebraic curve $C \subset \overline{Y}$ with $C \cap Y \neq \emptyset$ has $2g - 2 + (C \cdot Z) > 0$.

(C″) For X an algebraic variety any holomorphic mapping $f : X \to Y$ is algebraic.

- (B), (C), (C′), (C″) are related to (A) via well-known conjectures (cf. [ATY]).
- Given a variation of Hodge structure $(\mathcal{V}, F^{\bullet}; Y)$ (always assumed polarized) set $E^p = F^p/F^{p+1} = \mathrm{Gr}^p F^{\bullet}$ and let

$$\theta : TY \to \oplus \mathrm{Hom}(E^p, E^{p-1})$$

be the map induced by θ.

Theorem

θ generically injective $\implies$ (A), (B), and θ injective $\implies$ (C), (C′), (C″).

Conjecture

θ injective $\implies$ $K_{\overline{Y}}(\log Z), \Omega^1_{\overline{Y}}(\log Z)$ ample modulo Z; e.g., this means $K_{\overline{Y}}(\log Z)$ is semi-ample and any curve contracted by $|mK_{\overline{Y}}(\log Z)|$, $m \gg 0$, is in Z.

- One issue is the normal bundles of $Z_I \subset \overline{Y}$, where $Z_I = \cap_{i \in I} Z_i$. By an interesting formula these are related to the Hodge bundles of the limiting mixed Hodge structures. This will be discussed in the remark at the end of these talks.

- Geometric case: $X \xrightarrow{f} Y$ smooth fibration with X, Y quasi-projective and with $\mathbb{V} = R_f^n\mathbb{Q}$ ($\mathbb{V}_y = H^n(X_y, \mathbb{Q})$), $\mathrm{Var}\, f =$ rank of

$$T_y Y \to H^1(TX_y)$$

at a general point; here recall that for $x \in X_y$ the exact first connecting map in the cohomology sequence associated to

$$0 \to T_x X_y \to T_x X \to f^* T_y Y \to 0$$

gives $T_y Y \to H^1(TX_y)$ (Kodaira-Spencer map) and θ is the cup product with the Kodaira-Spencer class.

- θ induced by $TX_y \to \mathrm{Hom}(E_y^p, E_y^{p-1})$; injectivity is infinitesimal Torelli; here $E_y^p = H^{p,n-p}(X_y) \cong H^{n-p}(\Omega^p_{X_y})$.

(D) $\kappa(\overline{X}) \geqq \kappa(X_y) + \kappa(\overline{Y})$ where $\kappa =$ Kodaira dimension.

Theorem (Iitaka conjecture)

Assuming $\kappa(X_y) = \dim X_y$ *for general* $y \in Y$,
$\mathrm{Var}\, f = \dim Y \implies$ (D).

- $L = \overset{p}{\oplus} \det F^p$; $\omega =$ Chern form of L; canonical extensions $L_e \to \overline{Y}$ and ω_e.
- $\Phi : Y \to \Gamma\backslash D$ period mapping; will show that may assume Φ is proper; i.e., if monodromy T_i around Z_i is of finite order, then Φ extends across Z_i (as will be seen this is a theorem that uses a curvature argument).
- We will also see that $\omega_e \in L^1_{\mathrm{loc}}$ is a current representing $c_1(L_e)$ and for $\xi \in TY$, $\omega(\xi) = \|\Phi_*(\xi)\|^2$.

Theorem (BBT)

$\Phi(Y) \subset \Gamma\backslash D$ *is an algebraic variety* P *over which* $L \to P$ *is ample.*

Conjecture

$L_e \to \overline{Y}$ is semi-ample.

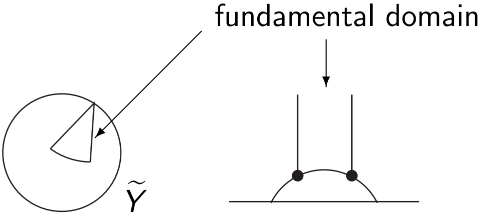
If true this would give a strong version of BBT and would open the door to defining Satake-Baily-Borel completions of arbitrary period mappings.

- Assume θ is injective; then $\omega =$ complete Kähler metric with curvature form $\Theta_Y(\xi, \eta)$ on a Zariski open in ξ, η space and finite volume; on $\widetilde{Y}$ universal cover of Y we have
$$\mathrm{Vol}(B_r(\widetilde{y}_0)) \geqq e^{\beta r}.$$
- Exhaustion function $\varphi = \Omega_{\check{D}}/\Omega_D\big|_{\widetilde{Y}}$ where
$$\varphi : \widetilde{Y} \to \mathbb{R}$$
with $\mathcal{L}(\varphi) = i\partial\overline{\partial} \log \varphi > 0$, and level sets are comparable to $\partial B_r(\widetilde{y}_0)$'s $\implies$ $\widetilde{Y} =$ Stein manifold (Shafarevich conjecture for Y's supporting a VHS).

Conjecture

$\widetilde{Y}$ can be realized as a bounded Stein variety in some $\mathbb{C}^N$.

Picture is



$$\eta = \frac{dz d\bar{z}}{(1-|z|^2)^2} \quad = \quad \frac{dx\, dy}{y^2} \quad \text{in} \quad \mathcal{H} \subset \mathbb{P}^1$$

- For Y complete the connected fibres of the Shafarevich map are subvarieties $W \subset Y$ with $\mathrm{im}\{\pi_1(W) \to \pi_1(Y)\}$ finite.

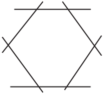
Conjecture

Assume θ is injective and for any index set I the N_i are linearly independent. Then there exist m_0 and $k_i > 0$ such that

$$mL_e - \sum k_i Z_i$$

is ample for $m \geqq m_0$.

- Example: $\dim Y = 2$ and $Z \subset \overline{Y}$ contracts to a cusp singularity; thus

$Z =$  $\|Z_i \cdot Z_j\| < 0.$

Then conjecture is true and the k_i are chosen to have $Z_i \cdot \sum k_j Z_j < 0$ for all i.

- Notations: VHS $= \{\mathcal{V}, \mathcal{F}^\bullet, Q, \nabla; Y\}$, fibre V:
  - $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_Y$ with $\nabla^2 = 0$;
  - $\mathcal{V}^\nabla = \mathbb{V} = \ker \nabla$ is a $\mathbb{Q}$-local system;
  - $\mathcal{F}^p \subset \mathcal{V}$ fibrewise defines a Hodge structure, fibre F^p;
  - $Q \in \mathcal{V}^\nabla$ defines a polarization in each fibre;
  - $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_Y$.

- Period domain $D = \{\text{PHS's } F^\bullet \subset V_{\mathbb{C}}\}$ satisfying HRI, HRII; compact dual $\check{D} = \{F^\bullet\}$ satisfying HRI

$$\begin{array}{ccc} D & \subset & \check{D} \\ \| & & \| \\ G(\mathbb{R})/G_0 & & G(\mathbb{C})/\text{parabolic} \\ & & \| \\ & & U/G_0. \end{array}$$

  where G_0 and U compact $(\mathrm{SL}_2(\mathbb{R}) \subset \mathrm{SL}_2(\mathbb{C})$, $G_0 = \mathrm{SO}(2)$, $U = \mathrm{SU}(2))$.
- $E^p \to Y$ have metrics with Chern connections and resulting curvatures Θ_{E^p}

$$\begin{aligned} \Theta_{E^p} &\in \mathrm{Hom}(E^p, E^p) \otimes A^{1,1}(Y) \\ &= A^{1,1}(\mathrm{Hom}(E^p, E^p)) \text{ (matrix valued (1,1)) form;} \end{aligned}$$

  Θ_{E^p} is skew-Hermitian.

Curvature form: $\Theta_{E^p}(e,\xi) = \Theta^{\alpha}_{\beta i \bar{j}} e_\alpha \bar{e}_\beta \xi^i \bar{\xi}^j$.

Interpretation: $\mathcal{O}_{\mathbb{P}E^p}(1)$ = line bundle with metric and Chern form ψ is (1,1) form on $\mathbb{P}E^p$: in vertical fibre ψ is standard (1,1) form on $\mathbb{P}E^p_y$ (Fubini-Study form); horizontal tangent space = $(\text{vertical})^\perp \cong T_y Y$ and ψ "given" by the curvature form.

Curvature formula:[2] $\theta^p : E^p \otimes T^{1,0}_Y \to E^{p-1}$ and Hermitian adjoint using HRII is $\theta^{p+1^*} : E^p \otimes T^{0,1}_Y \to E^{p+1}$. For $\xi, \eta \in TY$ and $u, v \in E^p$,

- $(\Theta_{E^p}(\xi,\eta)u, v) =$
  $(\theta^p(\xi)u, \theta^p(\eta)v) - (\theta^{p+1^*}(\eta)u, \theta^{p+1^*}(\xi)v)$;
- $\Theta_{E^\bullet} = -[\theta, \theta^*]$;
- $\Theta_{E^p} = -A_p \wedge {}^t\overline{A}_p + B_{p+1} \wedge {}^t\overline{B}_{p+1}$ (curvature matrix).

- Note that Θ_{E^p} has a sign on $\ker \theta^p$ and on $\ker \theta^{p+1*}$.

[2] Cf. [CM-SP] for the derivation of this formula.

Basic formula: $\nabla = \theta + \nabla_C + \theta^*$ on $\mathcal{V} \cong \oplus E^p$, $\nabla_C =$ Chern connection induced by $\nabla\big|_{V^{p,q}}$

$$\implies 0 = \nabla^2 = \nabla_C^2 + [\theta, \theta^*] \implies \Theta_E = -[\theta, \theta^*].$$

Application: Y complete $\implies$ any horizontal holomorphic section of $\mathbb{V}_\mathbb{C} \to Y$ has horizontal components.

Reason: $s = s_1 + \cdots + s_m$ type decomposition

$$\begin{aligned} \nabla s = 0 &\implies \theta_m \cdot s_m = 0 \\ &\implies \text{ curvature form on } s_m \text{ has a sign.} \end{aligned}$$

For any Hermitian vector bundle $E \to Y$ with holomorphic section e such that $(\Theta_E e, e) \leqq 0$ we have

$$\begin{aligned} &\partial\overline{\partial}(e, e) = (De, De) - (\Theta_E e, e) \\ \implies &\|e\|^2 \text{ is sub-harmonic } \implies \|e\|^2 = \text{ constant} \\ \implies &De = 0,\ (\Theta e, e) = 0. \end{aligned}$$

Applied to above gives $\nabla s_m = 0$, and continue.

Corollary

Any sub-bundle $\mathcal{V}' \subset \mathcal{V}$ fixed by ∇ and defined $/\mathbb{Q}$ is a sub-VHS ($\implies \mathcal{V} = \mathcal{V}' \oplus \mathcal{V}'^{\perp}$ giving semi-simplicity of monodromy).

Idea: Apply above to Plücker coordinate of $\wedge^m \mathcal{V}' \subset \wedge^m \mathcal{V}$.

- *tangent bundle*: is not a Hodge bundle but assuming θ is injective it is a sub-bundle of a Hodge bundle

$$TY \overset{\theta}{\hookrightarrow} \oplus \operatorname{Hom}(E^p, E^{p-1}) \, .$$

Given a VHS $(\mathcal{V}, \mathcal{F}^\bullet, \nabla; Y)$ we have a bundle $\mathfrak{g} \to Y$ of Lie algebras $\operatorname{End}(\mathcal{V})$ with Hodge decomposition $\mathfrak{g}^{p,q}$ and fibres

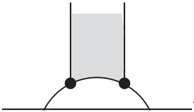
$$\mathfrak{g}^{-1,1} = \oplus \operatorname{Hom}(\mathcal{V}^{p,q}, \mathcal{V}^{p-1,q+1}).$$

Main observation: In the above for $\theta(TY)$, $\mathrm{Im}(\theta) \subseteq \mathrm{Ker}\,\theta$. This is integrability $\theta \wedge \theta = 0$; the image $\mathfrak{A} \subset \mathfrak{g}^{-1,1}$ is fibrewise an *abelian* Lie subalgebra.

- $\Theta_Y =$ curvature matrix for TY is given by $-\frac{1}{2}[\xi, \eta^*]$ where

$$\begin{aligned} \Theta_Y(\xi,\eta) &= \text{holomorphic bi-sectional curvature} \\ &= -([\xi,\eta^*],[\xi,\eta^*]) \leqq 0, \\ \Theta_Y(\xi) &= \text{holomorphic sectional curvature} \\ &= -\|[\xi,\xi^*]\|^2 < 0 \\ \implies \Theta_Y(\xi,\eta) &< 0 \ \text{ on a Zariski open in each fibre.} \end{aligned}$$

- Motto is: Period maps are "negatively curved"; property has many applications.
- Poincaré metric $\eta = \frac{dz \otimes d\bar{z}}{(1-|z|^2)^2}$ on $\Delta = \{|z| < 1\}$; Gauss curvature $K = -1$, invariant under $\mathrm{SL}_2(\mathbb{R})$.
- Induced Poincaré metric on Δ^* is $\psi = \frac{d\xi \otimes d\bar{\xi}}{|\xi|^2(\log|\xi|^2)^2} = \frac{dr\, d\theta}{r(\log r^2)^2}$; on circle $\gamma = |\xi| = r$ as $r \to 0$ the length $\ell(\gamma) \to 0$
  - area $\{|\xi| \leqq r\}$ is $\iint \frac{dr\, d\theta}{r(\log r^2)^2} \sim \int d\left(\frac{1}{\log r}\right) < \infty$
  - in $\begin{matrix} \mathcal{H} \subset \mathbb{P}^1 \\ \wr\| \\ \Delta^* \end{matrix}$



$\psi = \frac{dx^2 + dy^2}{y^2}$

- **Schwarz lemma:** Holomorphic $f : \Delta \to \Delta$, $f(0) = 0 \implies |f(z)| \leqq |z|$

$$\left.\begin{array}{ll} \implies & d_\Delta(f(z), f(z')) \leqq d_\Delta(z, z') \\ \implies & f^*\psi \leqq \psi \end{array}\right\} \begin{array}{l} f \text{ is distance decreasing in} \\ \text{Poincaré metric} \end{array}$$

- **Ahlfors lemma:** $f : \Delta \to M$ where M has a Hermitian metric with $(1,1)$ form ω and with holomorphic sectional curvatures $K \leqq -1$

$$\implies f^*\omega \leqq \psi.$$

- $\Phi : Y \to \Gamma \backslash D$ period mapping, assume immersion, curvature of $L \to Y$ gives

$$\omega = c_1(L) = \text{Kähler metric on Y with } K(\xi) \leqq -c > 0.$$

Note: ω has mixed signature on D; positive in the horizontal directions, negative in vertical ones for $G(\mathbb{R})/G_0 \to G(\mathbb{R})/K$.

Near a point of $Z = Y \backslash \overline{Y}$ we have taking one Z given by $z = 0$ so that locally around a point of Z we have

$$\Delta^* \times \Delta^{n-1} \hookrightarrow Y$$

$$\Longrightarrow \omega \leqq \frac{dz \wedge d\bar{z}}{|z|^2(-\log|z|)^2} + \text{(less singular terms)}$$

- parallel transport around circles $\gamma_m = (r = \frac{1}{m})$ give rise to monodromy T around γ_m;
- $y_m \to y_0 \in Z$,

$$\begin{aligned} d(y_m, \gamma_m y_m) &= d(g_m\bar{y}, Tg_m\bar{y}), \\ &\quad \text{where } y_m = g_m\bar{y} \text{ with } \bar{y} \in D = G(\mathbb{R})/G_0 \\ &= d(\bar{y}, g_m^{-1}Tg_m\bar{y}) \quad \text{(invariance of metric)} \\ &\longrightarrow 0 \text{ as } m \to \infty \end{aligned}$$

$$\Longrightarrow \text{eigenvalues } \lambda \text{ of integral matrix } T \text{ have absolute value } |\lambda| = 1$$

$$\Longrightarrow \lambda = e^{2\pi i p/q} \text{ (Kronecker).}$$

$\Longrightarrow$ **Monodromy theorem:** *Eigenvalues of T are roots of unity.*

- no monodromy

$$U^* \longrightarrow D$$

length of circle in D tends to zero

U = neighborhood in $\overline{Y}$ of a point of Z and $U^* = U \cap Y$ (take the case of one Z)

$\Longrightarrow$ circles shrink to a point of D (metric on D is complete);

$\Longrightarrow$ can extend Φ across Z;

$\Longrightarrow$ may assume $\Phi : Y \to \Gamma\backslash D$ is *proper* with image an analytic variety of finite volume;

$\Longrightarrow$ BBT gives that image is algebraic variety and $L \to \Phi(Y)$ ample.

- Analysis around $Z = \overline{Y} \backslash Y$; take one branch with monodromy T where $(T^k - I)^{m+1} = 0$; using orbifolds may assume $k = 1$ and $N = \log T$.

Theorem (monodromy weight filtration)

There exists a unique $W_k, -m \leqq k \leqq m$ *such that*

$$\left.\begin{array}{l} - \quad NW_k \to W_{k-2} \\ - \quad N^k : \mathrm{Gr}_{m+k} \xrightarrow{\sim} G_{m-k} \end{array}\right\} \textit{due to Schmid}$$

- *for* $v \in V_{\mathbb{C}}$ *we have*

$$v \in W_k \iff \|v\| \leqq (-\log|t|)^k$$

$\Longrightarrow$ *Since* $Nv = 0 \implies v \in W_{\leqq 0}$ *we have* $Tv = v \implies \|v\| \leqq$ *constant. Thus theorem of the fixed part and semi-singularity of monodromy hold in the general quasi-projective case.*

- $(V, W_\bullet, F^\bullet_\infty)$ where $F^\bullet_\infty = \lim_{t\to 0} \exp(tN) F^\bullet_0$ gives limiting mixed Hodge structure (LMHS).
- Wonderful fact is that the monodromy weight filtration given by Hodge norms.
- Cattani-Kaplan-Schmid analyzed the VHS over $\Delta^{*k} \times \Delta^j$ — in particular for the Chern forms $c_k(H)$ for a Hodge bundle H
  - $c_k(H)$ is bounded by Poincaré forms; $c_k(H)$ defines a closed current that represents $c_k(H_e)$;
  - we can multiply the $c_k(H)$ as though they are smooth forms.[3]

[3] Cf. [GG] for details.

Recall that ∇ has regular singular points; leads to Deligne extension $F_e^p \to \overline{Y}$ of Hodge bundles

$$
\begin{array}{ccc}
\overline{T}_Y(-\log Z) & \xhookrightarrow{\theta_e} & \Omega^1_{\overline{Y}}(\log Z) \otimes \mathrm{Gr}^\bullet \mathcal{F}_e) \\
\uparrow & & \\
TY & &
\end{array}
$$

- For $\Omega \in A^{n,n}(\overline{Y}, \log Z)$ induced by Hodge metrics the Ricci form $\mathrm{Ric}\,\Omega$ defines a positive closed $(1,1)$ current that is in $L^1_{\mathrm{loc}}(\overline{Y})$ and whose cohomology class restricted to Y gives $c_1(K_{\overline{Y}})$.

- Locally

$$\Omega = h\Big(\bigwedge_{j=1}^{n}\Big(\frac{i}{2}\Big)dz_j \wedge d\bar{z}_j\Big), \quad h > 0$$

$$\mathrm{Ric}\,\Omega = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h$$

where

  - h has logarithmic singularities;
  - Ω has Poincaré singularities and $\Omega > 0$ where θ is injective.

$\implies$ $c_1(K_{\overline{Y}}(\log Z)) \geqq 0$ and $c_1(K_{\overline{Y}}(\log Z)) > 0$ on Zariski open in $\overline{Y}^*$.

- Similar considerations apply to $T_{\overline{Y}}(-\log Z)$; this follows from the next bullet and leads to (A), (B) above.
- *Relation between* $\Theta_Y(\xi, \eta)$ *and* $\Theta_Y(\xi)$ (cf. [BKT]).

Lemma

Suppose $\Theta_Y(\xi, \eta) \leqq 0$ *and* $\Theta_Y(\xi) \leq c < 0$*. Then there exists* ξ_0 *such that* $\Theta_Y(\xi_0, \eta) \leqq -c/2$*.*

As a corollary, $\Theta_Y(\xi, \eta) < 0$ on a Zariski open set in $TY \times TY$. In particular the Chern form ψ of $\mathcal{O}_p(1)$ on $P = \mathbb{P}TY$ has $\psi \geqq 0$ and $\psi > 0$ on a Zariski open set. Using

$$H^0(Y, \mathrm{Sym}^m \Omega^1_Y) \cong H^0(P, \mathcal{O}_P(m))$$

this implies that if Y is complete, then Ω^1_Y is big and nef. In general we get the same result for $\Omega^1_{\overline{Y}}(\log Z))$.

- Very brief sketch of the proof of the lemma:
  - Choose ξ_0 where $\Theta_Y(\xi)$ is a maximum.
  - For $\Theta_Y(\xi_0 + t\eta)$ at $t = 0$ the first t-derivative is zero and second derivative is $\leqq_0$.
  - By making clever use of the identities on the curvature tensor of a Kähler metric conclude that for some η_0 we have $\Theta_Y(\xi_0, \eta_0) \leqq -c/2$.
- Corollary of Ahlfors lemma: $\Delta(R) \xrightarrow{f} Y$ and $\|f'(0)\| = 1 \implies R \leqq \mathbb{R}_0 < \infty$

  $\implies$ hyperbolicity of Y if θ is injective.
- Algebraicity results from Bishop theorem and finite volume of graph of Φ restricted to $\Delta^{*k} \times \Delta^{n-k}$'s.
- Recently much work on arithmetic consequences of negative curvature; e.g., [JL]:

"THEOREM 1.1 (Main Result, I) *Let $A \subset k = \overline{\mathbb{Q}}$ be a finitely generated subring and let $\mathcal{X}$ be a finite type A-scheme such that $\mathcal{X}_k$ is a quasi-projective variety over k which admits a quasi-finite complex-analytic period map. Then the following statements are equivalent:*

(1) *For every finitely generated subring $A' \subset k$ containing A, the set $\mathcal{X}(A')$ is finite (resp. not Zariski-dense) in $\mathcal{X}(k)$).*

(2) *For every finitely generated integral domain B containing A, the set $\mathcal{X}(B)$ is finite (resp. not Zariski-dense in $\mathcal{X}(\overline{Frac(B)})$) (where $\overline{Frac(B)}$ is a choice of algebraic closure of Frac(B)).*

In other words, for varieties admitting a quasi-finite period map, finiteness of $\mathcal{O}_{K,S}$-points (where K ranges over all number fields and S ranges over all finite collections of finite places of K) implies finiteness of A-points for all $\mathbb{Z}$-finitely generated integral domains A of characteristic zero, and a similar statement (which requires substantially deeper input) holds for non-Zariski-density of rational points. Both the finiteness and non-density results require input from Hodge theory. Arguably, the novel technical result in our proof of Theorem 1.1 is Theorem 3.7."

- **Iitaka conjecture:** *$X \xrightarrow{f} Y$ and*
  - *X_y general type;*
  - *$\mathrm{Var}\, f = \dim Y$;*

$\implies \kappa(X) \geqq \kappa(X_y) + \kappa(Y)$.

- Assume X, Y and general X_y are smooth

$$K_X \text{``}=\text{''} K_{X/Y} \otimes f^* K_Y$$

("$=$" means that the correction from singular X_y's will be concentrated over a proper subvariety of Y and corresponding Hilbert polynomial will have degree $< \dim Y$)

$$\Longrightarrow \quad H^0(K_X) \text{“}\cong\text{”} H^0(K_{X/Y} \otimes f^*K_Y) \longleftarrow H^0(K_{X/Y}) \otimes H^0(K_Y);$$

$$\wr\| \qquad H^0(f_*K_{X/Y})$$

$\Longrightarrow$ many sections of $H^0(f_*K_{X/Y}) \Longrightarrow$ many sections of $H^0(K_X)$ (assuming $h^0(K_Y) \neq 0$);[4]

$\Longrightarrow$ need positivity of $f_*K_{X/Y}$ = Hodge bundle $V^{n,0}$ where $\dim X_y = n$.

- Strong local Torelli: $T_y Y \to \mathrm{Hom}(E^n, E^{n-1})$ generically injective
  - $\Longrightarrow$ $E^n = V^{n,0}$ has positivity of the curvature form
  - $\Longrightarrow$ result we want.
- Hodge metric on $H^0(K_{X_y})$ is given by

$$(\psi, \eta) = \int_{X_y} \psi \wedge \overline{\eta}.$$

[4] Actually should use $\omega_{X/Y}$ instead of $K_{X/Y}$.

- This induces metric in $(\mathbb{P}H^0(K_{X_y})^*, \mathcal{O}(1)) := M_y$, and as above positivity of curvature form for $H^0(K_{X_y})$'s $\to Y \leftrightarrow$ positivity of curvature form for M_y.
- In general only have $H^0(K_{X_y}^{\otimes m}) \neq 0$ for $m \gg 0$.
- The new geometric idea (Kawamata-Vieweg) is to use sections ψ of $K_{X_y}^{\otimes m'} s \to Y$ to produce cyclic branched coverings $\widetilde{X}_{y,\psi} \to X_y$ together with $\widetilde{\psi} \in H^0(K_{\widetilde{X}_{y,\psi}})$.
- Differential geometrically, for $\psi \in H^0(K_{X_y}^{\otimes m})$

$$\|\psi\|^{2/m} = \|\widetilde{\psi}\|^2 = \int (\psi \wedge \overline{\psi})^{2/m}$$

is a *Finsler metric* on $H^0(K_{X_y}^{\otimes m})$; induces a metric in $\widetilde{\mathcal{O}}_m(1) \to \mathbb{P}H^0(K_{X_y}^{\otimes m})$.

- Varying X_y, ψ as above gives family of $\widetilde{\psi} \in H^0(K_{\widetilde{X}_{y,\psi}})$'s and curvature form from the VHS gives the Chern form of $\widetilde{\mathcal{O}}_m(1)$.
- As for singularities these at most contribute terms whose h^0 grows like a Hilbert polynomial of lower degree.
- After a significant amount of technical argument one has a proof of Iitaka (delicate estimates on Chern forms of Hodge bundles are needed; Cattani-Kaplan-Schmid plus refinement by Kollár).
- Use of curvature via $L^2 - \overline{\partial}$ methods applied to singular metrics is by now a vast subject; cf. papers by Paun and Zuo plus many others.

- **Remark:** The "geometry at infinity" of a period mapping is still a work in progress. Along $Z_I := \bigcap_{i \in I}$ we have a generalized period mapping

$$\Phi_I : Z_I^* \to \Gamma_I \backslash D_I$$

where D_I is a period domain for mixed Hodge structures and Φ_I is given by the LMHS's.
The first level is given by the mapping

$$\Phi_I^0 : Z_I^* \to \Gamma_I^0 \backslash D_I^0$$

to the associated graded pure Hodge structures.
For the next level given

$$0 \to A \to B \to C \to 0$$

where A, C are pure Hodge structures where $\mathrm{wt}(C) = \mathrm{wt}(A) + 1$ we have

$$B \leftrightarrow \mathrm{Ext}^1_{\mathrm{MHS}}(C, A) = \text{compact, complex torus.}$$

It turns out that the induced mapping

$$\Phi_I^1 : Z_I^* \to \left\{ \begin{array}{c} \text{level 1} \\ \text{extension} \\ \text{data} \end{array} \right\}$$

maps to an abelian subvariety of the above $\mathrm{Ext}^1_{\mathrm{MHS}}$'s. The basic relation between the geometry along Z_I and the geometry normal to Z_I is expressed by the formula on a Φ_I^0-fibre

$$(*) \qquad \Phi_I^{1,*}(\mathcal{L}) + \sum_i k_i[Z_i] = 0, \quad k_i > 0$$

where $\mathcal{L} \to \mathrm{Ext}^1_{\mathrm{MHS}}(C, A)$ is a line bundle that is ample on the image of Φ_I^1.

- The next level of extension data maps to a $\prod \mathbb{C}^*$, and on $\Phi_I^0 = \Phi_I^1 = \Phi_I^2 =$ constant the remaining extension data is constant.

Example

If $\dim Y = 2$ and Φ^0 is constant, then for connected Z, $(*)$ gives that

$$\|Z_i \cdot Z_j\| < 0$$

and so Z contracts to a normal singularity.

A significant missing piece to the story is to make full use of $(*)$ as part of a more complete understanding of Φ at infinity.

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