Positivity and Vanishing Theorems

Phillip Griffiths*

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Abstract

Existence theorems are a central part of algebraic geometry. These results frequently involve linear problems where positivity assumptions are used to prove existence of solutions by establishing the vanishing of obstructions to that existence. Beginning with Riemann (algebraic curves), Picard (algebraic surfaces) and continuing into more recent times (Lefschetz, Hodge, Kodaira-Spencer and many others since this work) it has come to be understood that the vanishing theorems are intimately related to the topology of algebraic varieties.

What remains is the case that although the results are about algebraic varieties, analytic tools are needed to establish them. Moreover the property of positivity also appears in other aspects where analytic methods are needed, an example being the proof of the Iitaka conjecture which is central in the classification of algebraic varieties. The purpose of these lectures is to present, sometimes from an historical perspective, some of the principal aspects of the theory.

Outline

I. The classical case (Picard [PS], Lefschetz [Le], Kodaira-Spencer [GH])

II. Analytic methods involving singular metrics and curvatures ([Ko1], [Ko2], Andreotti-Vesentini [AV], Hormander [H], Demailly [De],...)

III. Consequences of positivity of the Hodge bundles (Vieweg [V1], [V2], [V3], Vieweg-Zuo [VZ],...) and the freeness conjecture

Lecture I: Topology and vanishing theorems

It is general knowledge that there is a close relationship between the topological properties of algebraic varieties and vanishing theorems for sheaf cohomology ([K1]). Here we will try to present the essence of this relationship from a somewhat different perspective than is generally done. We are not seeking the most general statements, some of which will be taken up in the second lecture. Rather we are interested in understanding the conceptual aspects of the relationship.

Notations (terminology to be elaborated on below; general references are [De] and [GH])

- X is an n-dimensional, irreducible smooth algebraic variety;
- $L \to X$ is an ample line bundle;
- $Y \subset X$ is the smooth divisor of a holomorphic section $s \in H^0(X, L)$.

We will generally not distinguish between a holomorphic vector bundle and the sheaf of its holomorphic sections. One may think of $X \subset \mathbb{P}^N$, $L = \mathcal{O}_X(m)$ and Y is given by intersecting X with a general degree m hypersurface.

- $\omega = c_1(L) \in H^2(X, \mathbb{Q})$ is the cohomology class dual to the cycle given by $Y \subset X$;
- Ω_X^p is the sheaf of holomorphic p-forms on X, $\Omega_Y^n = K_X$ is the canonical bundle;
- a *Hodge structure of weight r* on $H^r(X, \mathbb{Q})$ is given by

$$H^r(X, \mathbb{C}) = \bigoplus^{p+q=r} H^{p,q}(X) \text{ where } \overline{H}^{p,q}(X) = H^{q,p}(X).$$

The basic facts we shall use are ([GH])

HT: The cohomology of a smooth algebraic variety has a functorial Hodge structure, and

$$H^{p,q}(X) \cong H^q(X, \Omega_X^p).$$

Here HT stands for *Hodge theory*.

Basic results about the topology of algebraic varieties:

LT: The restriction map

$$H^r(X, \mathbb{Q}) \to H^r(Y, \mathbb{Q}) \text{ is } \begin{cases} \left\{ \begin{matrix} \text{an isomorphism} \\ \text{for } r \leqq n-2 \end{matrix} \right\} \\ \\ \left\{ \begin{matrix} \text{injective for} \\ r = n-1 \end{matrix} \right\} \end{cases}$$

HL: $H^{n-r}(X, \mathbb{Q}) \overset{\omega^r}{\xrightarrow{\sim}} H^{n+r}(X, \mathbb{Q})$ is an isomorphism.

Here LT stands for *Lefschetz theorem* and HL for what is usually called *Hard Lefschetz*. The first result is true with $\mathbb{Z}$ coefficients.

Basic vanishing theorems ([De] and [GH])

KVT: $H^q(X, K_X + L) = 0$ for $q > 0$.

KVT stands for *Kodaira vanishing theorem*. We will sometimes use the additive notation for line bundles

$$K_X + L = K_X \otimes_{\mathcal{O}_X} L.$$

In the second lecture we will give a proof of the KVT using singular metrics.

KAN: $H^q(X, \Omega^p_X(L)) = 0$ for $p + q > n$.

Here KAN stands for Kodaira-Akizuki-Nakano. The KVT is the case $p = n$ of KAN. Using Kodaira-Serre duality, KAN is equivalent to

$$H^q(X, \Omega^p_X(L^{-1})) = 0 \text{ for } p + q < n, \tag{1.1}$$

where L^{-1} is the dual line bundle to L

$$L^{-1} = \mathrm{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X).$$

Summary of the relations between topology and vanishing theorems: Assuming HT these are

$$\text{LT} \Rightarrow \text{KAN} \tag{1.2}$$

$$\text{KAN} \Rightarrow \text{LT} + \text{HL}. \tag{1.3}$$

Historical remarks: As is well known, for algebraic curves (compact Riemann surfaces) these results are due to Riemann. Perhaps lesser known is that for algebraic surfaces and in somewhat different form these results are essentially consequences of the work of Picard and Poincaré (cf. [PS]). The essential steps in the argument in this case are

(i) $H^q(X,\mathbb{Q}) \to H^q(Y,\mathbb{Q})$ is an isomorphism for $q=0$ (i.e., Y is connected), and is injective for $q=1$;

(ii) there is an inclusion

$$H^0(X,\Omega^1_X) \hookrightarrow H^1(X,\mathbb{C});$$

and

(iii) using this inclusion the map

$$H^0(X,\Omega^1_X) \xrightarrow{\omega} H^3(X,\mathbb{C})$$

is injective.

Picard's proof of (i) was geometric using what is now called a *Lefschetz pencil* $|Y_t|$ given by intersecting $X \subset \mathbb{P}^N$ with a general pencil of hyperplanes. The classical references here are [PS] and [Le]. Picard's proof of (ii) involved his construction of the identity component $\mathrm{Pic}^\circ(X)$ of the Picard variety.

A modern proof of (ii) is deceptively simple: For $\varphi \in H^0(X, \Omega^1_X)$ by Stokes theorem

$$0 = \int_X d(\varphi \wedge \overline{d\varphi}) = \int_X d\varphi \wedge \overline{d\varphi}$$

which is non-zero if $d\varphi \neq 0$. Thus $d\varphi = 0$ and the result follows.

The key result is (iii). For Y a general member of the Lefschetz pencil, Picard defined two subgroups:

- $E \subset H_1(Y, \mathbb{Q})$, the *vanishing cycles* given by the kernel of $H_1(Y, \mathbb{Q}) \to H_1(X, \mathbb{Q})$; and
- $I \subset H_1(Y, \mathbb{Q})$, the *invariant cycles* given by the subgroup invariant under monodromy when Y varies in a Lefschetz pencil.

Using the Poincaré complete reducibility theorem (a Hodge-theoretic result), Picard argued that the Jacobian variety $J(Y)$ is isogeneous to a *direct sum* of two polarized abelian varieties, one corresponding to E and the other to I. In particular,

$$E \cap I = (0). \tag{1.4}$$

To prove (iii), given $\gamma \in H_1(X, \mathbb{Q}) \cong I \subset H_1(Y, \mathbb{Q})$ by Poincaré duality it will suffice to produce $\Delta \in H_3(X, \mathbb{Q})$ such that the intersection number

$$(\gamma, Y \cap \Delta) \neq 0.$$

By the complete reducibility theorem we may find $\delta \in I \subset H_1(Y, \mathbb{Q})$ such that $(\gamma, \delta) \neq 0$. Then varying Y_t in a Lefschetz pencil the family of 1-cycles given by the $\delta_t \in H_1(Y_t, \mathbb{Q})$ traces out the desired 3-cycle Δ.

With the benefit of hindsight one may say that Picard was able to mobilize just enough about Hodge theory to give a proof of HL in the case of surfaces. The Lefschetz argument for HL was incomplete; essentially because he didn't have a proof of (1.4).

Proofs of (1.3) **and** (1.2) (due to Kodaira-Spencer and given in [GH]): We will prove the dual form (1.1) of KAN. The argument is by induction on dimension; thus we assume the result for Y. Use will be made of the long exact cohomology sequences arising from the exact sequences of sheaves

$$\begin{aligned} &0 \to \Omega^p_X(L^{-1}) \to \Omega^p_X \to \Omega^p_X\big|_Y \to 0 \\ &0 \to \Omega^{p-1}_Y(L^{-1}) \to \Omega^p_X\big|_Y \to \Omega^p_Y \to 0. \end{aligned}$$

Using local coordinates $x_1, \dots, x_n$ where Y is given by $x_n = 0$, in the first sequence for a φ given by

$$\varphi = f(x_1, \dots, x_n) dx_i \wedge \cdots \wedge dx_{i_p}$$

we set $x_n = 0$ but do not set $dx_n = 0$. In the second sequence, for

$$\psi = g(x_1, \dots, x_{n-1}) dx_i \wedge \cdots \wedge dx_{i_p}$$

the right-hand map on ψ is given by setting $dx_n = 0$. If ψ is in the kernel, then

$$\psi = g(x_1, \dots, x_{n-1}) dx_1 \wedge \cdots \wedge dx_{i_{p-1}} \wedge dx_n$$

and dx_n gives a section of the co-normal bundle

$$N^*_{Y/X} = L^{-1}\big|_Y.$$

The cohomology sequences of the above give

$$\begin{tikzcd}
\to H^q(X,\Omega_X^p(L^{-1})) \arrow[r] & H^q(X,\Omega_X^p) \arrow[r, "\alpha"] & H^q(X,\Omega_X^p\big|_Y) \arrow[r] \arrow[dl, "\beta"'] & H^{q+1}(X,\Omega_X^p(L^{-1})) \to \\
H^q(Y,\Omega_Y^{p-1}(L^{-1})) \arrow[r] & H^q(Y,\Omega_X^p\big|_Y) \arrow[r, "\gamma"] & H^q(Y,\Omega_X^p) \arrow[r] & H^{q+1}(Y,\Omega_Y^{p-1}(L^{-1})).
\end{tikzcd}$$

Note that β is the identity map. The assertions LT and KAN for Y are

- $\gamma \circ \beta \circ \alpha$ is an isomorphism for $p+q \leqq n-2$ and is injective for $p+q=n-1$; and
- $H^q(Y,\Omega_Y^{p-1}(L^{-1}))=0$ for $p+q<n$.

It follows from the cohomology diagram that KAN holds for X. Conversely and more visibly, if KAN holds then for $p+q<n$ all the $H^q(X,\Omega_X^p(L^{-1}))$ and $H^q(Y,\Omega_Y^{p-1}(L^{-1}))$ groups for $p+q<n$ are zero, which then implies LT.

KAN $\Rightarrow$ HL[†]

Since basically it makes rigorous Picard's original argument given in [PS], we will give the proof for surfaces and then indicate how it may be extended to the general case.

The central issue is: What exact sheaf sequences will have multiplication by ω appearing in the resulting exact cohomology sequences? One answer is that we want to have a Gysin map $G : H^1(Y, \mathbb{C}) \to H^3(X, \mathbb{C})$ appear, and this occurs for the coboundary map in the exact sheaf sequences when there is a Poincaré residue map R. For surfaces we have

$$0 \to \Omega_X^2 \to \Omega_X^2(Y) \xrightarrow{R} \Omega_Y^1 \to 0$$

[†] It may not be so well known that Hard Lefschetz is a consequence of vanishing theorems, plus of course the existence of a Hodge structure on cohomology and Kodaira-Serre duality.

where if Y is given locally in coordinates x, y for X by $y = 0$

$$R(f(x,y)dx \wedge dy/y) = \left(\frac{1}{2\pi i}\right) f(x,0)dx.$$

Using $H^1(X, \Omega_X^2(Y)) = H^1(X, K_X + L) = 0$ we obtain the commutative diagram

$$\begin{tikzcd}
H^0(X,\Omega_X^2) \arrow[r] & H^0(X,\Omega_X^2(Y)) \arrow[r] & H^0(Y,\Omega_Y^1) \arrow[r, "G"] & H^1(X,\Omega_X^2) \arrow[r] & 0 \\
& & H^0(X,\Omega_X^1) \arrow[u, "j"] \arrow[ur, "\omega"'] & &
\end{tikzcd}$$

For $\varphi \in H^0(X, \Omega_X^1)$, a standard consequence of Poincaré duality about the Gysin map G gives

$$\int_X G \circ j(\varphi) \wedge \overline{\varphi} = \int_X \varphi \wedge \overline{\varphi} \wedge \omega = \int_Y \varphi \wedge \overline{\varphi}.$$

Since by Kodaira-Serre duality $h^{1,0}(X) = h^{2,1}(X)$, this implies that ω is an isomorphism, which is HL in this case.
This argument used two special facts:

(i) for n=forms we have

$$\Omega^n_X(Y) = \Omega^n_X(\log Y);$$

and

(ii) for a non-zero holomorphic n form ψ on an n-dimensional algebraic variety Z we have

$$\int_Z \psi \wedge \overline{\psi} \neq 0.$$

For (i) in the general case one uses the cohomology sequences arising from the exact sheaf sequences

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega^p_X \arrow[r] & \Omega^p_X(\log Y) \arrow[r, "R"] \arrow[d] & \Omega^{p-1}_Y \arrow[r] & 0 \\
 & & \Omega^p_X(Y) \arrow[d, "\alpha"] & & \\
 & & \Omega^p_Y(L) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The only non-obvious map is α. Here the point is the natural isomorphism

$$\Omega^p_X(Y)/\Omega^p_X(\log Y) \cong \Omega^p_Y \otimes N_{Y/X}.$$

We will illustrate this when $n = 3$, $p = 2$ from which the general case should be clear. In local coordinates x_1, x_2, y where Y is given by $y = 0$, for

$$\varphi = f(x_1, x_2, y)\frac{dx_1 \wedge dx_2}{y} + g_1(x_1, x_2, y)\frac{dx_1 \wedge dy}{y} + g_2(x_1, x_2, y)\frac{dx_2 \wedge dy}{y}$$

the map is

$$\alpha(\varphi) = f(x_1, x_2, 0)dx_1 \wedge dx_2 \otimes \partial/\partial y.$$

One checks directly that $\alpha(\varphi)$ transforms properly under a coordinate change

$$\begin{cases} x_i = g_i(x_1, x_2, y) & i = 1, 2 \\ y = h(x_1, x_2, y)y, & h(x_1, x_2, 0) \neq 0. \end{cases}$$

Positivity: The original KVT was the following:

Let X be a compact, complex manifold and $L \to X$ a holomorphic line bundle whose Chern class is represented via de Rham's theorem by a positive $(1,1)$ form ω. Then

$$H^p(X, K_X + L) = 0 \text{ for } p > 0. \tag{1.5}$$

Locally

$$\omega = \left(\frac{i}{2}\right) \sum_{i,j} h_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

where $h_{ij} = \overline{h}_{j\bar{i}}$ is a positive definite Hermitian matrix. We write this as $\omega > 0$. The KVT was used to prove the *Kodaira embedding theorem (KET):*

A positive line bundle $L \to X$ is ample.

There are two steps in Kodaira's proof of the above theorems.

(i) show that there is a Hermitian metric h in $L \to X$ such that

$$\omega = \left(\frac{i}{2\pi}\right)\Theta_h$$

where Θ_h is the curvature form of the metric; and

(ii) if $L \to X$ is a Hermitian line bundle such that

$$\left(\frac{i}{2\pi}\right)\Theta_h > 0,$$

then using (1.5) and the technique of blowing up Kodaira shows that for $L \to X$ as above

(1.6) $\qquad$ *$L \to X$ is ample.*

We will use singular metrics to give proofs of (1.5) and (1.6) in the next lecture.

There have been many extensions of KVT and KAN. For algebro-geometric purposes it is desirable to have assumptions that (a) are numerical, and (b) that behave well under birational morphisms. Especially noteworthy is the

KV: *Kawamata-Vieweg vanishing theorem (cf. [De] and [L]):* (1.5) *holds under the assumption that* $L \to X$ *is big and nef.*

Here *big* means that

$$h^0(X, L^m) = Cm^n + \cdots, C > 0;$$

i.e., for $m \gg 0$ the linear system $|mL|$ gives a rational map to an n-dimensional subvariety of a projective space; *nef* means that for any curve $C \subset X$

$$\deg(L\big|_C) = L \cdot C \geqq 0.^{\ddagger}$$

‡It is a result of Kleiman that for L nef and for any k-dimensional subvariety $Y \subset X$

$$\langle c_1(L)^k, [Y] \rangle \geqq 0.$$

Both of the properties hold for $L' = f^*L$ for a generically finite morphism $f : X' \to X$.

One of the traditional proofs of Kawamata-Vieweg uses KAN and the so-called cyclic covering trick which we will discuss following a parenthetical note.

The role of Hodge theory: The thesis of this lecture has been the close connection between vanishing theorems and topology, a connection that is made using Hodge theory. As posed by Kollár and others, one may ask "How much of Hodge theory is really needed?" One answer is

The KVT follows from the surjectivity of the natural map

$$H^q(X, \mathbb{C}) \twoheadrightarrow H^q(X, \mathcal{O}_X), \qquad q \geqq 0 \tag{1.7}$$

induced by the inclusion of sheaves $\mathbb{C}_X \hookrightarrow \mathcal{O}_X$.

The question is not purely academic, as the surjectivity of the map (1.7) makes sense for X a singular variety, and one may ask if assuming (1.7) leads to a class of singular varieties that have particularly nice topological properties. As shown by Steenbrink, Kollár-Kovács and others, the answer is that Du Bois singularities have this property and for these an important part of the local invariant cycle theorem holds.

Here we shall give the argument for the KVT under the additional assumption that $L \to X$ is very ample. As will be seen both the Hodge-theoretic result (1.7) and an additional topological argument will be used in the proof. Although in the case at hand it can be avoided, we will use the Leray spectral sequence as it provides a conceptual framework for the argument in the general case.

The setup is

- $L \to X$ is as above and $s \in H^0(X, L^2)$ is a section with smooth divisor $D = \{s = 0\} \subset X$;
- $Y \subset L$ is the smooth graph of $s^{1/2}$; i.e.,
$$Y = \{(x, \ell) : x \in L, \ell \in L_x \text{ and } \ell^2 = s(x)\};$$
- $Y \xrightarrow{p} X$ is the 2:1 covering branched over D and with sheet interchange
$$\tau : Y \to Y$$
where $Y/\tau = X$.

We next decompose the direct image sheaves of $\mathcal{O}_Y$ and C_Y into the ± 1 eigenspaces as

(i) $p_* \mathcal{O}_Y = \mathcal{O}_X \oplus L^{-1}$;

(ii) $p_* \mathbb{C}_Y = \mathbb{C}_X \oplus \mathbb{C}_X[-1]$.

The reason for (i) is that for any vector bundle holomorphic functions defined in open set of the total bundle space and that are linear in the fibres give sections of the *dual* vector bundle over the projection of the open set to the base. In (ii) the -1 eigensheaf $\mathbb{C}_X(-1)$ of $p_* \mathbb{C}_Y$ may be considered as a local system on $X \backslash D$ that has monodromy -1 around D. The two Leray spectral sequences degenerate and give a commutative diagram

$$\begin{tikzcd}
H^q(Y, \mathcal{O}_Y) \arrow[r, phantom, "\cong"] & H^q(X, \mathcal{O}_X) \arrow[r, phantom, "\oplus"] & H^q(X, L^{-1}) \\
H^q(Y, \mathbb{C}_Y) \arrow[r, phantom, "\cong"] \arrow[u, two heads] & H^q(X, \mathbb{C}_X) \arrow[r, phantom, "\oplus"] \arrow[u, two heads] & H^q(X, \mathbb{C}_X[-1]). \arrow[u, two heads]
\end{tikzcd}$$

By Hodge theory the vertical arrows are surjective; therefore the KVT will follow from

$$H^q(X, \mathbb{C}_X[-1]) = 0 \text{ for } q < n.$$

This is a purely topological result and we refer to [K1] for a proof. Remark that the essential point is

$$H^{2n-q}(X \backslash D, \mathbb{C}) = 0 \text{ for } q < n,$$

which is the same as the LT

$$H_q(X, D) = 0 \text{ for } q < n.$$

Note: An alternate simpler argument goes as follows: For $L \to X$ ample and $s \in H^0(X, L)$ a section with a smooth divisor $Y \subset X$, we have

$$0 \to L^{-1} \xrightarrow{s} \mathcal{O}_Y \to \mathcal{O}_Y \to 0.$$

By LT the top maps

$$\begin{tikzcd} H^q(X,\mathbb{C}) \arrow[r] \arrow[d] & H^1(Y,\mathbb{C}) \arrow[d] \\ H^q(X,\mathcal{O}_X) \arrow[r] & H^q(Y,\mathcal{O}_Y) \end{tikzcd}$$

are isomorphisms for $q \leqq n-2$ and injective for $q = n-1$. Then the exact cohomology sequence of the above sheaf sequence gives $H^q(X, L^{-1}) = 0$ for $q < n$.

In case $L \to X$ is just ample, for some $m \gg 0$ we may find a section $s \in H^0(X, L^m)$ with a smooth divisor and construct an m-fold cyclic covering $Y \to X$ by taking the graph of $s^{1/m}$; the remainder of the argument proceeds essentially as the one given above.

Appendix to Lecture I

For completeness we list the general properties associated to cyclic coverings. Recall our notations

- $L \to X$ is an ample line bundle over a smooth algebraic variety X;
- $s \in H^0(X, L^m)$ is a section whose zero locus gives a smooth divisor $D \subset X$.

Note: The theory extends in a straightforward manner to the case when D is a reduced normal crossing divisor.

- $Y \xrightarrow{p} X$ is the cyclic branched covering associated to $s^{1/m}$, and $R \subset Y$ is the smooth ramification locus of the mapping p;
- $Y \xrightarrow{p} X$ is a Galois covering with group μ_m the m^{th} roots of unity.

Note: The following properties of $Y \xrightarrow{p} X$ may be deduced from the local normal form

$$(y, x_2, \ldots, x_n) \to (y^m, x_2, \ldots, x_n)$$

of the map. Locally $s(x_1, x_2, \ldots, x_n) = x_1^{1/m}$.

- the direct image

$$p_* \mathcal{O}_Y \cong \bigoplus_{i=0}^{m-1} L^{-i}; \tag{A.1}$$

  here the right-hand side is given by the eigenspace decomposition of the action of μ_m on $p^{-1}(\mathcal{U})$ where $\mathcal{U} \subset X$ is an open set;
- $p^* \Omega_X^p(\log D) \cong \Omega_Y^p(\log R)$;
- the direct image

$$p_* \Omega_Y^p \cong \bigoplus_{i=0}^{m-1} \Omega_X^p(\log D) \otimes L^{-i}, \tag{A.2}$$

  where again the right-hand side is the eigenspace decomposition under the action of μ_m;

- For similar reasons the direct image

$$(A.3) \qquad p_*\Omega^p_Y(\log R) \cong \bigoplus_{i=0}^{m-1} \Omega^p_X(\log D) \otimes L^{-i}.$$

- From Hodge theory (cf. [EV] and the references cited there)

$$(A.4) \qquad H^r(Y \backslash R, \mathbb{C}) \cong \bigoplus_{p+q=r} H^q(Y, \Omega^p_Y(\log R)).$$

- Since $X \backslash D$ is affine, and hence a Stein manifold, the same is true of the unramified covering $Y \backslash R$ of $X \backslash D$, from which we may infer that

$$H^r(Y \backslash R, \mathbb{C}) = 0 \text{ for } r > n.$$

Then from $(\mathrm{A}.4)$ and the degeneration of the Leray spectral sequence for p we have
(A.5)

$$H^q(X, \Omega^p_X(\log D) \otimes L^{-i}) = 0 \text{ for } p+q > n,\ 0 \leqq i \leqq m-1.$$

- Taking $p = n$ and using $\Omega^n_X(\log D) \cong \Omega^n_X \otimes L^m$ we obtain from the case $i = m-1$ in $(\mathrm{A}.5)$ that

$$H^q(X, \Omega^n_X \otimes L) = 0 \text{ for } q > 0.$$

This is the KVT. The full KAN vanishing theorem may be obtained from $(\mathrm{A}.5)$ by an inductive argument on dimension as was done in the above notes for Lecture 1. The subtlety here is that $(\mathrm{A}.4)$ is a result in mixed Hodge theory; one way or another to obtain the full KAN vanishing theorem where $L \to X$ is only assumed to be ample, the use of mixed Hodge theory is necessary.

The constant local system $\mathbb{C}_Y$ gives a Gauss-Manin connection on $\mathbb{C}_Y \otimes_{\mathbb{C}} \mathcal{O}_Y \cong \mathcal{O}_Y$. The group μ_m acting on Y preserves the local system and Gauss-Manin connection; using (A.1) it descends to give a local system on $X \backslash D$. Taking the 1-eigenspace of this action, considerations using the local normal form of $p : Y \to X$ show that this local system extends to give a local system on X with a logarithmic singularity along D. Thus, essentially by decreeing that

$$\nabla(s^{1/m}) = 0$$

we obtain a local system (L^{-1}, ∇) on X giving rise to a twisted de Rham complex

$$\begin{aligned} 0 \to (p_* \mathbb{C}_Y)^1 \to L^{-1} \to \Omega^1_X(\log D) \otimes L^{-1} \\ \to \cdots \to \Omega^n_X(\log D) \otimes L^{-1} \to 0. \end{aligned}$$

The Hodge-de Rham spectral sequence degenerates at E_1 with the result being for general i that the eigenspace decomposition of the Hodge structure on $H^r(Y, \mathbb{Q})$ under the action of μ_m may be realized down on X by direct sums of the groups $H^q(X, \Omega^p_X(\log D) \otimes L^{-i})$, i.e., by the Hodge-de Rham complex on X twisted by the local system L^{-i}.
An important special case is when $L = K_X$, i.e., when X has an ample canonical bundle. Classically Hodge theory is frequently used to study the moduli of X, this being quite effective when K_X is very ample and local Torelli holds in the form of the injectivity of the map

$$H^1(X, T_X) \to \mathrm{Hom}(H^0(X, \Omega^n_X),\ H^1(X, \Omega^{n-1}_X).$$

In general, however only $|mK_X|$ is very ample for $m \gg 0$, and we may ask if Hodge theory can still be used to study moduli.

Basically the question is:

> *Is there a Hodge-theoretic interpretation of $H^0(X, K_X^m)$?*

An affirmative answer is given by the work of Kawamata, Vieweg, Zuo and others (cf. [VZ] and [Z]). This will be discussed further in the third lecture in connection with the proof of the Iitaka conjecture. Here we will give a brief preview of this story. This preview will also explain why we singled out the eigenspace for $i = 1$ for the local systems on the L^{-i}'s above. This discussion will be in two steps.

Step 1: *Realization of $H^0(X, K_X^m)$ as part of a Hodge structure.*

Specifically we have

- $H^0(X, K_X^m)$ is the 1-eigenspace of the action of μ_m on $H^0(Y, K_Y)$.

Proof: From (A.2) and the degeneration of the Leray spectral sequence we have (here $L = K_X$ and $D = K_X^m$)

$$\begin{aligned} H^0(Y, \Omega_Y^n) &\cong \bigoplus_{i=1}^{m} H^0(X, \Omega_X^n(\log D) \otimes K_X^{-i}) \\ &= \bigoplus_{i=1}^{m} H^0(X, K_X \otimes K_X^m \otimes K_X^{-i}), \end{aligned}$$

which gives the result.

Step 2: *How this Hodge structure varies with X.*

This is subtle and we refer to [VZ] for details. The idea is that

- this is an inclusion $H^0(X, K_X^m) \subset H^0(Y, \Omega_Y^n)$;
- as (X, s) vary we get a deformation of Y whose tangent is in $H^1(X, T_Y(-\log R))$;
- under the map $H^1(Y, T_Y(-\log R)) \to H^1(Y, T_Y)$ we obtain a map

$$H^0(X, K_X^m) \xrightarrow{\xi} H^1(\Omega_Y^{n-1}); \text{ and}$$

- finally we decompose the image under the action of μ_m to obtain the expression for the variation of $H^0(X, K_X^m)$.

This will be further discussed in Lecture 3.

Lecture II: Analytic methods involving singular metrics and curvature

So how is one going to prove vanishing theorems and use those to establish existence theorems? In this lecture we will try to illustrate one way how one may answer these questions. We begin with an example that shows how proving existence necessitates proving a vanishing result and how the use of singular metrics may be used to obtain the desired vanishing.

Example: We begin with a preliminary lemma, and for this we work in a germ of a neighborhood of the origin $\{0\}$ in $\mathbb{C}^n$. For

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p), \qquad \alpha_i \in \mathbb{N}$$

we denote by $I_{\boldsymbol{\alpha}}$ the ideal in $\mathcal{O}_{\mathbb{C}^n,\{0\}}$ generated by all monomials $z_1^{\beta_1} \cdots z_p^{\beta_p}$ such that

$$\sum_i (\beta_i + 1)/\alpha_i > 1.$$

For example the maximal ideal $\mathfrak{m}_{\{0\}}$ is the case $p = n$ and all $\alpha_i = 1$, in which case the above condition is equivalent to at least one $\beta_i > 1$. For

$$\varphi_0 = \log\left(|z_1|^{\alpha_1} + \cdots + |z_p|^{\alpha_p}\right)$$

we have the calculus

Lemma

$I_{\boldsymbol{\alpha}} = \{f \in \mathcal{O}_{\mathbb{C}^n,\{0\}} : \int e^{-2\varphi_0}|f|^2 < \infty\}$.

Here for ϵ sufficiently small the integral is over an ϵ-ball around the origin.

The import of the lemma is that the position of $f(z)$ relative to monomial ideals in $\mathcal{O}_{\mathbb{C}^n,\{0\}}$ may be detected using L^2 conditions involving singular, plurisubharmonic weight functions of the above form. Similar methods work for other ideals.

Now we let $L \to X$ be a holomorphic line bundle over a compact complex manifold. Given $x_0 \in X$ we want to construct a global section $s \in H^0(X, L)$ with $s(x_0) \neq 0$. Let h be an arbitrary smooth Hermitian metric in $L \to X$, φ a function on X that near x_0 looks like the φ_0 above for the case of the maximal ideal $\mathfrak{m}_{x_0} \subset \mathcal{O}_{X,x_0}$ plus a C^∞ function and is smooth away from x_0, and choose a smooth Hermitian metric for L with volume form Ω on X.

Let s_0 be a holomorphic section of L defined over a neighborhood $\mathcal{U}$ of x_0 and with $s_0(x_0) \neq 0$ and let ρ be a C^∞ bump function equal to 1 near x_0 and compactly supported in $\mathcal{U}$. The ρs_0 is a global C^∞ section of $L \to X$ with

$$\psi = \overline{\partial}(\rho s_0) = \overline{\partial}\rho \cdot s_0 = 0 \text{ near } x_0.$$

We have

- $\overline{\partial}\psi = 0$
- $\int_X e^{-2\varphi}|\psi|^2\Omega < \infty$.

Here $|\psi|^2$ is the norm of ψ using the metrics on X and L. Suppose we can find a C^∞ section σ of $L \to X$ such that

- $\overline{\partial}\sigma = \psi$
- $\int e^{-2\varphi}|\sigma|^2\Omega < \infty$.

Then σ is holomorphic near x_0, and consequently

$$s = \rho s_0 - \alpha$$

is a global holomorphic section of $L \to X$ with $s(x_0) \neq 0$. We conclude that *Solving the $\overline{\partial}$-equation*

$$(1) \qquad \overline{\partial}\sigma = \psi \text{ where } \overline{\partial}\psi = 0$$

with the singular weight function $e^{-2\varphi}$ as above implies the existence of global holomorphic sections of $L \to X$ that are non-vanishing at a given point.

It will be seen that solving the $\overline{\partial}$-equation (1) with the L^2 weight conditions as above can be done when the curvature form ω_h of the metric in $L \to X$ is sufficiently positive.

Specifically, for $\mathrm{Ric}\, X$ the curvature form in the canonical bundle K_X, (1) will have a solution provided that

$$\omega_h + \mathrm{Ric}\, X + \left(\frac{i}{2\pi}\right) \partial\overline{\partial}\varphi > 0 \tag{2}$$

is a positive (1,1) current (terminology to be explained below). If $\omega_h > 0$, then replacing L by L^m which replaces ω_h by $m\omega_h$, one may prove the existence of sections of $L^m \to X$ that generate all fibres (i.e., L is free or semi-ample). Similar arguments for $\mathfrak{m}_x^2$ and $\mathfrak{m}_{x_1} \otimes_{\mathcal{O}_X} \mathfrak{m}_{x_2}$ lead to a proof of the Kodaira existence theorem. Thus it comes down to solving the $\overline{\partial}$-equation (1).

A priori estimates: *How is one going to prove the existence of solutions to* (1)*?* It is an inhomogeneous linear *overdetermined* PDE with the *integrability conditions* $\overline{\partial}\psi = 0$. The classical method is to proceed in two steps:

(i) solve (1) in the weak sense;

(ii) show that a weak solution is smooth (regularity).

Here (i) means that there exists an L^2 section σ of $L \to X$ such that for any smooth L-valued (0,1) form η we have

$$(\sigma, \overline{\partial}^* \eta) = (\psi, \eta) \tag{3}$$

where $(\ ,\ )$ is the global L^2 inner product and $\overline{\partial}^*$ is the adjoint operator to $\overline{\partial}$. We will not discuss (ii) but will focus on (i).

The key to solving (i) will be to show that there is a pointwise positive algebraic operator A such that the *a priori estimate*

$$(4) \qquad \|\overline{\partial}\gamma\|^2 + \|\overline{\partial}^*\gamma\|^2 \geqq (A\gamma,\gamma) > 0$$

holds.
Here $\ker\overline{\partial}$ is the closure in L^2 of $\overline{\partial}$ (smooth forms).

Claim: *If* (4) *holds, then we can solve the equation* (1).

Proof (sketch): We have

$$L^2 = (\ker\overline{\partial}) \oplus (\ker\overline{\partial})^{\perp}.$$

Let θ be a smooth form and $\theta = \theta_1 + \theta_2$ the corresponding decomposition. Since $(\ker\overline{\partial})^{\perp} \subset \ker\overline{\partial}^*$, by duality we have $\overline{\partial}^*\theta_2 = 0$. Then using Cauchy-Schwarz

$$|(\psi,\theta)|^2 = |(\psi,\theta_1)|^2 \leqq \int_X (A^{-1}\psi,\psi)\Omega \cdot \int_X (A\theta_1,\theta_1)\Omega.$$

Applying (4) to the second term gives

$$\int_X (A\theta_1, \theta_1) \leqq \|\overline{\partial}\theta_1\|^2 + \|\overline{\partial}^*\theta_1\|^2 = \|\overline{\partial}^*\theta\|^2.$$

Combining both inequalities there is a positive constant $C = C(\psi)$ such that

$$|(\psi, \theta)|^2 \leqq \left(\int_X (A^{-1}\psi, \psi)\Omega\right)\left(\|\overline{\partial}^*\theta\|^2\right) \leqq C\|\overline{\partial}^*\theta\|^2.$$

This holds for every θ, so there is a well-defined linear form

$$\overline{\partial}^*\theta \to (\psi, \theta).$$

By the Hahn-Banach theorem there exists $\sigma \in L^2$ such that for every θ

$$(\psi, \theta) = (\sigma, \overline{\partial}^*\theta) = (\overline{\partial}\sigma, \theta).$$

This gives a weak solution to (1), and then the ellipticity of the $\overline{\partial}$-operator (or rather of its associated Laplacian) leads to the proof of the existence of a solution to (1) in the usual sense. $\square$

What is the idea behind the above argument?

If we have a complex

$$\cdots \to C^{i-1} \xrightarrow{\delta_{i-1}} C^i \xrightarrow{\delta_i} C^{i+1} \to \cdots$$

where the C^i are finite dimensional vector spaces with inner products, then there is an associated Laplacian

$$\Delta_i = \delta_i^* \delta_i + \delta_{i-1} \delta_{i-1}^*$$

with harmonic space

$$\mathcal{H}^i = \ker \Delta_i = (\ker \delta_i) \cap (\ker \delta_{i_1}^*).$$

By linear algebra the inclusion $\mathcal{H}^i \hookrightarrow C^i$ induces an isomorphism of the cohomology groups

$$H^i(C^\bullet) = \ker \delta_i / \mathrm{im}\, \delta_i \cong \mathcal{H}^i.$$

But an a priori estimate

$$(\Delta_i \alpha, \alpha) = \|\delta_i \alpha\|^2 + \|\delta_{i-1}^* \alpha\|^2 \geqq C\|\alpha\|^2$$

gives that the harmonic space $\mathcal{H}^i = 0$.

Positivity and priori estimates: The relationship between curvature and vanishing theorems originated with Bochner [Bo] and under suitable curvature assumptions was applied to show the vanishing of certain de Rham cohomology groups on a compact Riemann manifold M. The basic formula that was used is the Weitzenböck identity

$$\Delta \alpha = \sum_i (\nabla_i^* \nabla_i + \nabla_i \nabla_i^*)\alpha + R(\alpha)$$

where α is a differential form, the ∇_i are covariant derivatives and R is a curvature operator.

This gives

$$(\Delta\alpha, \alpha) = \|\nabla\alpha\|^2 + (R\alpha, \alpha),$$

which implies that if $R(\alpha)$ is a pointwise positive operator, then the harmonic space is zero. It was this argument that Kodaira adapted to the case of the $\overline{\partial}$-cohomology of holomorphic line bundles over a compact complex manifold. In the Kähler case (and only in this case) the Riemann geometry and complex geometry align, meaning that the Riemannian (Levi-Civita) connection and Chern connection coincide. For (n, p)-forms the curvature operator simplifies in that the curvature of the manifold drops out leaving only the curvature of the line bundle, and the Bochner argument then gives the KVT.

We shall now briefly recall the essential steps in the calculations that lead to the KAN vanishing theorem.

Curvature and the basic identity (cf. [De] and [GH])

- X is a compact, complex manifold.
- $E \to X$ is a holomorphic vector bundle. We will identify E with the sheaf $\mathcal{O}_X(E)$ of holomorphic sections of $E \to X$.
- $L \to X$ is a holomorphic line bundle.

The main example of a holomorphic vector bundle we shall consider is

$$E = \Omega_X^p(L) = \wedge^p T^* X \otimes L.$$

- $A^{0,q}(X, E)$ denotes the space of global, C^∞ E-valued $(0, q)$ forms on X.

If $z_1, \ldots, z_n$ are local holomorphic coordinates on X and $e_1, \ldots, e_r$ is a local holomorphic frame for $E \to X$, then $\psi \in A^{0,q}(X, E)$ is locally

$$\psi = \sum_{|I|=q} f_I^\alpha(z, \bar{z}) d\bar{z}^I \otimes e_\alpha(z)$$

where $I = (i_1, \ldots, i_q)$, $d\bar{z}^I = d\bar{z}^{i_1} \wedge \cdots \wedge d\bar{z}^{i_q}$ and $f_I^\alpha(z, \bar{z})$ is a C^∞ function.

- We have

$$\cdots \to A^{0,q}(X,E) \xrightarrow{\overline{\partial}} A^{0,q+1}(X,E) \to \cdots, \quad \overline{\partial}^2 = 0$$

  and the Dolbeault cohomology of this complex will be denoted by

$$H^{0,q}_{\overline{\partial}}(X,E) = \frac{\ker\{A^{0,q}(X,E) \xrightarrow{\overline{\partial}} A^{0,q+1}(X,E)\}}{\overline{\partial} A^{0,q-1}(X,E)}.$$

In local coordinates, for ψ as above

$$\overline{\partial}\psi = \sum_{|I|=q} \overline{\partial} f_I^{\alpha}(z,z) \wedge d\bar{z}^I \otimes e_\alpha(z)$$

where

$$\overline{\partial} f = \sum_i \partial_{\bar{z}_i} f(z,\bar{z}) d\bar{z}^i.$$

Although we will not make explicit use of it, lurking behind the scenes is the Dolbeault theorem

$$H^{0,q}_{\overline{\partial}}(X, E) \cong H^q(X, \mathcal{O}_X(E))$$

which connects the usual sheaf cohomological aspects of algebraic geometry to global solutions of equations arising from the $\overline{\partial}$-operator. One of the points of these lectures is to illustrate how in complex algebraic geometry one may frequently use global analytic methods in place of the more standard algebraic and cohomological tools. As we have noted even if one prefers the latter, to get some of the deepest results analysis becomes necessary. See [De] for many illustrations of this.

- A Hermitian metric h in $L \to X$ is given in terms of a local holomorphic frame $e(z)$ by

$$|e(z)|^2 = h(z, \bar{z})$$

where h is a positive C^∞ function. The curvature

$$\Theta_{L,h} = \overline{\partial}(h^{-1}\partial h) = \overline{\partial}\partial \log h$$

is independent of the holomorphic frame. The line bundle is (metrically) *positive* if there is an L such that the real (1,1) form

$$\omega_h := \left(\frac{i}{2\pi}\right)\Theta_{L,h} := \left(\frac{i}{2\pi}\right)\sum_{i,j}\partial_{\bar{z}_j}\partial_{z_i}(\log h)dz^i \wedge d\bar{z}^j$$

is positive, written $\omega_h > 0$.

For later use, if we write

$$h = e^{-2\varphi},$$

then

$$\omega_h = \left(\frac{i}{\pi}\right) \partial\overline{\partial}\varphi$$

so that $\omega_h > 0$ is the same as φ pluri-subharmonic.

- Given a holomorphic vector bundle $E \to X$ with smooth Hermitian metric $(\ ,\ )$ in the fibres there is a unique *Chern connection*

  $$D : A^0(X, E) \to A^1(X, E) = A^{1,0}(X, E) \oplus A^{0,1}(X, E)$$

  that satisfies
  - $D = D' + D''$ where $D'' = \overline{\partial}$;
  - $d(e, e') = (De, e') + (e, De')$ where $e, e' \in A^0(X, E)$.

We refer to [De] and [GH] for the basic properties of the Chern connection and its curvature.

- We assume given $L \to X$ and Hermitian metrics in this bundle and in TX. This defines metrics in $L \otimes \wedge^p T^*X \otimes \wedge^q \overline{T^*X}$. By integration over X the operators

$$D' : A^{p,q}(X,L) \to A^{p+1,q}(X,L), \quad D'^2 = 0$$
$$D'' : A^{p,q}(X,L) \to A^{p,q+1}(X,L), \quad D''^2 = 0$$

have adjoints

$$D'^* : A^{p,q}(X,L) \to A^{p-1,q}(X,L)$$
$$D''^* : A^{p,q}(X,L) \to A^{p,q-1}(X,L)$$

and resulting Laplacians

$$\Delta' = D'^* D' + D' D'^*$$
$$\Delta'' = D''^* D'' + D'' D''^* = \overline{\partial}^* \overline{\partial} + \overline{\partial}\overline{\partial}^*.$$

As noted above the harmonic spaces for Δ'' are isomorphic to the Dolbeault cohomology groups.

- The metric in TX is Kähler if any of the equivalent conditions
  - the Chern connection is equal to the Riemannian one;
  - the (1,1) form ω_X associated to the metric is closed;
  - at each point of X the metric osculates to second order to the the flat metric on $\mathbb{C}^n$.

  The third condition means that at each point there are local holomorphic coordinates $z^1, \ldots, z^n$ so that

  $$g_{i\bar{j}}(z, \bar{z}) = \delta_{ij} + [2]$$

  where $g_{i\bar{j}} = (\partial_{z^i}, \partial_{\bar{z}^j})$ and [2] are quadratic terms in the coordinates. It implies that any first order identity that holds in $\mathbb{C}^n$ also holds on the manifold. The basic ones are those on page 111 in [GH] and (4.5) and (4.6) in [De].

- The pointwise operator

$$\omega_X : L \otimes \wedge^p T^*X \otimes \wedge^q \overline{T^*(X)} \to L \otimes \wedge^{p+1} T^*X \otimes \wedge^{q+1} \overline{T^*X}$$

  of wedging with ω_X has an adjoint

$$\Lambda_X : L \otimes \wedge^p T^*X \otimes \wedge^q \overline{T^*X} \to L \otimes \wedge^{p-1} T^*X \otimes \wedge^{q-1} \overline{T^*X}$$

  and by a linear algebra computation the basic commutation relation

$$[\omega_X, \Lambda_X] = (p + q - n)\,\mathrm{Id} \tag{5}$$

  is satisfied.
- Now we assume that $L \to X$ is positive and choose for our Kähler metric the one whose associated (1,1) form is the Chern form ω_h. Then using (5) one obtains the identity (4.6) in [De]

$$\Delta'' = \Delta' + (p + q - n)\,\mathrm{Id}\,.$$

The proof of this identity may be done by reducing it to first order identities and then checking these in $\mathbb{C}^n$ which then gives the result on X by the osculating principle above. Since

$$(\Delta'\alpha, \alpha) = (D'\alpha, D'\alpha) + (D'^*\alpha, D'^*\alpha) \geqq 0,$$

we arrive at the *basic a priori estimate*

$$(6) \quad \|\overline{\partial}\psi\|^2 + \|\overline{\partial}^*\psi\|^2 > (p+q-n)\|\psi\|^2, \qquad p+q>n$$

for $\psi \in A^{p,q}(X, L)$. Using the identification of harmonic spaces with the $H^{p,q}_{\overline{\partial}}(X, L)$, this gives a proof of the Kodaira-Akizuki-Nakano vanishing theorem.

Singular weight functions and the Nadel vanishing theorem:

The basic idea here appears in the example at the beginning of this lecture. Given a holomorphic line bundle $L \to X$ with a smooth metric h we will use singular functions φ defined on X and with two properties:

(i) the locally defined functions f that satisfy

$$\int e^{-2\varphi}|f|^2 < \infty$$

give a coherent sheaf $\mathcal{I}_\varphi \subset \mathcal{O}_X$ of ideals in X; and

(ii) integration of φ against C^∞ functions on X defines a distribution and we have the equation of currents

$$\omega_h + \left(\frac{i}{2\pi}\right)\partial\overline{\partial}\varphi \geqq \psi \tag{7}$$

where ψ is a C^∞ positive (1,1) form on X.

We think of $e^{-2\varphi}$ as defining a singular metric in $L \to X$ whose singularities will detect special analytic properties of holomorphic sections of the bundle.

Theorem (Nadel)

Under the above assumptions,

$$H^q(X, (K_X + L) \otimes \mathcal{I}_\varphi) = 0 \;\; \textit{for} \quad q > 0.$$

Corollary

If $\omega_h > 0$, then for $m \geqq m_0$

$$H^q(X, L^m \otimes \mathcal{I}_\varphi) = 0 \;\; \textit{for} \;\; q > 0.$$

As we saw in the example this gives that a positive line bundle $L \to X$ is ample. The import of the above is that using singular metrics

$$e^{-2\varphi} h$$

in $L \to X$ gives more flexibility than is possible using only traditional smooth metrics.For our purposes the weight functions φ are those that are locally of the form

$$\varphi = \varphi' + \varphi''$$

where φ' is plurisubharmonic (psh) and φ'' is C^∞. We refer to [De] for a general discussion of psh functions and currents.

Here we will consider the case

- $\varphi' \in L^1_{\mathrm{loc}}$;
- $\left(\frac{i}{2\pi}\right) \partial\overline{\partial}\varphi \geqq 0$ in the sense of currents.

The first condition implies that φ' defines a distribution, and the second condition is that for all $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$

$$\sum_{i,j} \frac{\partial^2 \varphi'}{\partial z^i \partial \bar{z}^j} \lambda_i \overline{\lambda}_j$$

is a positive measure; i.e., for η a real, compactly supported, non-negative C^∞ function

$$\left\langle \varphi', \left(\sum_{i,j} \frac{\partial^2 \eta}{\partial z^i \partial \bar{z}^j} \lambda_i \overline{\lambda}_j \right) \right\rangle \geqq 0.$$

Example: $\varphi' = \left(\frac{\alpha}{2}\right)\log(|f_1|^2 + \cdots + |f_p|^2)$, $\alpha > 0$ is such a function. In the terminology used in [De] these are psh functions with analytic singularities.
We will not prove (i) here. For many if not most applications to algebraic geometry the φ''s of the above type, where the result may be seen directly, suffices; the general argument is in [De].

For the proof of the NVT there are two steps:

(a) establish the Dolbeault theorem $H^q(X, (K_X + L) \otimes \mathcal{I}_\varphi) \cong H^{n,q}_{\bar\partial}(X, L \otimes \mathcal{I}_\varphi)$ in the current context of currents, and
(b) show that $H^q_{\bar\partial}(X, (K_X + L) \otimes \mathcal{I}_\varphi) = 0$ for $q > 0$.

Both of these come down to existence theorems for solutions to the $\overline{\partial}$-equation in L^2, the first being local and the second being global. And in each case the argument is essentially the same as the one given above in the global case and for smooth metrics, the central point being again to have an a priori estimate. This is a consequence of the assumption (7).

Lecture III

Among the "big" results in the classification of algebraic varieties one may single out the following two as being particularly important.

Proof of the Iitaka conjecture (explained below). This leads to the understanding that there are three types of basic "building blocks" for algebraic varieties:

(i) varieties X of general type, i.e., those whose Kodaira dimension $\kappa(X) = \dim X$;

(ii) varieties X with $\kappa(X) = 0$; this includes Calabi-Yau varieties, abelian varieties, etc.;

(iii) varieties X with $\kappa(X) = -\infty$; includes the Fano varieties, in particular those having a smooth divisor $D \in |-K_X|$ (del Pezzo surfaces, etc.).

Here building block means that any X is birationally an iterated fibration whose general fibre is one of the types (i), (ii), (iii). We recall that the Kodaira dimension $\kappa(X) = d$ if

$$h^0(X, mK_X) = Cm^d + \cdots , \quad C > 0.$$

If $h^0(X, mK_X) = 0$ for all m, we set $\kappa(X) = -\infty$.
The other big result is the minimal model program (MMP) which we shall not discuss.

The proof of the Iitaka conjecture (Kawamata, Vieweg, Kollár [Ka], [V1], [V2], [V3], [K2] with refinements and extensions by many others) uses the ingredients

(a) Hodge theory; specifically the positive curvature properties of the Hodge line bundle (HLB);

(b) the cyclic covering trick, already encountered in Lecture I; and

(c) extensive use of standard birational geometry (base change, resolution of singularities, etc.).

So what is the Iitaka conjecture? Let

$$f : X \to Y$$

be a morphism between smooth projective varieties and $X_y = f^{-1}(y)$ a general fibre. Then the conjecture, now a result, is

$$\kappa(X) \geqq \kappa(X_y) + \kappa(Y). \tag{1}$$

We shall discuss (1) in what turns out to be the essential case for establishing the general conjecture, namely when

(A) $\kappa(X_y) = \dim X_y \qquad (X_y \text{ is of general type})$

(B) $\mathrm{Var}\, f = \dim Y$.

Here the second assumption means intuitively that the map from the parameter space Y to the moduli space $\mathcal{M}$ of a general fibre X_y is generically locally 1-1. In technical terms it means that for a general point $y \in Y$ the Kodaira-Spencer map

$$\rho_y : T_y Y \to H^1(X_y, TX_y)$$

is 1-1. Here one may think of ρ_y as the differential of the map $Y \to \mathcal{M}$ at the general point y.

The result (1) may be deduced from the following result:
Under the assumptions (A) and (B), for $m \gg 0$

$$\det(f_*\omega^m_{X/Y}) \text{ is big and nef.} \tag{2}$$

This means that the line bundle $\det(f_*\omega^m_{X/Y})$ has lots of sections, and later in this lecture we will try to explain intuitively how this leads to sufficiently many sections of $K_X^{\otimes k} \to X$, $k \gg 0$, to give the result.
In fact to better understand the original intuition, under additional assumptions we shall first prove a much stronger result than (2). This argument will illustrate the essence of (a) above, i.e., how Hodge theory enters the picture. The additional assumptions we make are

$$\begin{cases} \bullet \text{ the fibres } X_y \text{ are all smooth;} \\ \bullet \text{ a suitable version of local Torelli holds.} \end{cases} \tag{3}$$

Here the second assumption means the following: Using

$$H^0(X_y, K_{X_y}) = H^{n,0}(X_y) \subset H^n(X_y, \mathbb{C})$$

we obtain a local mapping from Y to the Grassmannian of $h^{n,0}(X_y)$ planes in the locally constant vector space $H^n(X_y, \mathbb{C})$, and local Torelli here means that this mapping should have injective differential.[§]

Theorem

Under the assumption (3) *above*

$$\det(f_*\omega_{X/Y}) \text{ is ample.}$$

[§] The usual meaning of local Torelli is that locally in moduli the way in which the Hodge decomposition on $H^n(X_y)$ varies determines X_y. Here we only consider the end piece $H^{n,0}(X_y) \cong H^0(\Omega^n_{X_y})$ of the Hodge decomposition.

Proof (sketch): $f_*\omega_{X/Y}$ is a vector bundle with fibre $H^0(X_y, \Omega^n_{X_y})$ the holomorphic n-forms on X_y. This Hodge bundle has a metric given by the inner product

$$(\varphi, \psi) = c_n \int_{X_y} \varphi \wedge \overline{\psi}$$

where $\varphi, \psi \in H^0(X_y, \Omega^n_{X_y})$ and c_n is a suitable constant. The metric then induces one in the Hodge line bundle

$$\Lambda := \det f_*\omega_{X/Y}.$$

The theorem will follow from Kodaira's theorem if we show that

(4) *the curvature form of $\Lambda \to Y$ is positive.*

This is a special case of the general principle: The curvature forms of the Hodge bundles $R^q_{f_*}\Omega^p_{X/Y}$ have special sign properties.[¶]

To prove (4) we let $\mathcal{U} \subset Y$ be a neighborhood of a point and

$$\Phi : \mathcal{U} \to \mathrm{Grass}(h^{n,0}, \mathbb{C}^b)$$

the map described above where all $H^n(X_y, \mathbb{C})$ for $y \in \mathcal{U}$ are identified with a fixed $\mathbb{C}^b$. Then the crucial curvature computation is that for $\xi \in T_y Y$

$$\left(\frac{i}{2}\right) \Theta_\wedge(\xi) = \|\Phi_*(\xi)\|^2.$$

[¶] The general curvature properties of Hodge vector bundles appear in many places in the literature; some recent references are [GG], [Z], and [LSZ].

This says that the curvature of the Hodge line bundle is a *first order* (not second as is usually the case for curvatures) invariant that measures the size of the first order variation of $H^{n,0}(X_y)$ in the locally constant vector space $H^n(X_y, \mathbb{C})$. $\square$

The assumption that local Torelli holds is too strong; this is frequently but not always the case (the general question of finding conditions where it holds is one of interest). A better algebro-geometric assumption is that the X_y are canonically embedded, so we now assume

$$(5) \qquad \begin{cases} \bullet \quad \text{the } X_y \text{ are smooth and } \rho_y \text{ is everywhere 1-1} \\ \bullet \quad \text{the canonical maps } \varphi_{I_{X_y}} : X_y \to \mathbb{P}H^0(X, K_{X_y})^* \\ \quad\;\; \text{are embeddings.} \end{cases}$$

Theorem

Under the assumptions (5), *for* $m \gg 0$

$$\det(f_*\omega_{X/Y}^{\otimes m}) \text{ is ample.}$$

If we only assume that ρ_y is generically 1-1, i.e., that $\mathrm{Var}\, f = \dim Y$, then the result becomes

$$\det\left(f_*\omega_{X/Y}^{\otimes m}\right) \text{ is big and nef.}$$

There seems to be little known about the stable base locus of the linear systems $|k \det(f_*\omega_{X/Y}^{\otimes m})|$ for $k, m \gg 0$. For example there is the question/conjecture

Q: *Is* $\det\left(f_*\omega_{X/Y}^{\otimes m}\right)$ *free for* $m \gg 0$*?*

Proof of the theorem: The argument is particularly interesting due to the blend of differential-geometric and algebro-geometric inputs that go into it. We shall use the exact sequence

$$0 \to R_m \to S^m(f_*\omega_{X/Y}) \xrightarrow{\mu} f_*\omega_{X/Y}^{\otimes m}. \tag{6}$$

The mapping μ is given by the pointwise multiplication

$$S^m H^0(X_y, K_{X_y}) \to H^0(X_y, K_{X_y}^{\otimes m})$$

of global sections of $K_{X_y} \to X_y$. For $m \gg 0$ the kernel R_m of μ is a vector bundle given by the degree m defining equations of the canonical model of the X_y's. We denote by C_m the image of μ so that denoting by

$$H = f_*\omega_{X/Y}$$

the Hodge vector bundle the sequence (6) becomes

$$\begin{cases} 0 \to R_m \to S^m H \to C_m \to 0 \\ 0 \to C_m \to f_*(\omega_{X/Y}^{\otimes m}) \to D_m \to 0. \end{cases} \tag{7}$$

The ideas are the following:

- H is semi-positive, and thus so is $S^m H$ (semi-positivity is a general property of globally generated vector bundles);
- since positivity increases on quotient bundles (another general property), C_m is semi-positive;
- $f_*(\omega_{X/Y}^{\otimes m})$ is semi-positive (discussed below), and as a consequence we may infer that

$$\det(f_*\omega_{X/Y}^{\otimes m}) = \det C_m \otimes \det D_m$$

  is semi-positive and is *strictly* positive if $\det C_m$ is positive;
- the positivity of C_m is increased from that induced from $S^m H$ by an amount that reflects the "twisting" of the subspaces $R_{m,y} \subset S^m H_y$;

and finally

- since for $m \gg 0$ the subspace $R_{m,y} \subset S^m H_y$ determines X_y, we will see that the assumption that the Kodaira-Spencer maps are 1-1 will imply there is sufficient twisting to give that $\det C_m$ is positive.

In order to give the details we need the following

Differential-geometric preliminaries (cf. [GG] and [De]: For a Hermitian vector bundle $E \to M$ the curvature form is defined for each $x \in M$, $e \in E_x$ and $\xi \in T_x M$ by

$$\Theta_E(e,\xi) = \left\langle (\Theta_E e, e)\,, \xi \wedge \bar{\xi} \right\rangle . \tag{8}$$

The bundle is *semi-positive* if $\Theta_E(e,\xi) \geqq 0$, *positive* if strict inequality holds (for e, ξ non-zero). Semi-positivity is a fairly common property of vector bundles, strict positivity much less so (cf. [GG]). We will abbreviate (8) by $\Theta_E \geqq 0$.

We note that $\Theta_E \geqq 0$ implies $\Theta_{\det E} \geqq 0$. Moreover, if M is a curve and everywhere along the curve we have $\Theta_E \geqq 0$ but do *not* have $\Theta_{\det E} > 0$, then $E \to M$ is flat (a positive semi-definite Hermitian matrix whose trace is zero is itself equal to zero).

Next let

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of holomorphic vector bundles. Then there is a *second fundamental form* of $S \subset E$ that measures how much the Chern connection on E fails to leave S invariant. This leads to a C^∞ operator

$$B \in A^{1,0}(M, \mathrm{Hom}(Q, E))$$

such that for $e \in Q_x \cong S_x^\perp \subset E_x$ and $\xi \in T_x M$ we have

$$\Theta_Q(e, \xi) = \Theta_E(e, \xi) + \left\langle (Be, Be), \xi \wedge \overline{\xi} \right\rangle . \tag{9}$$

In words, the amount by which the curvature increases in quotient bundles is measured by the size of the second fundamental form of S in E.
From this and the above we conclude that if $\Theta_E \geqq 0$ and we do not have $\Theta_{\det Q} > 0$, then there is a curve $C \subset M$ such that

- $E\big|_C$ is flat;
- $S\big|_C \subset E\big|_C$ is a flat sub-bundle.

We now apply these general results to the situation at hand. As previously noted we have an Hermitian metric in H with $\Theta_H \geqq 0$. This gives $\Theta_{S^m H} \geqq 0$, and then $\Theta_{C_m} \geqq 0$.

If we do *not* have $\Theta_{\det C_m} > 0$, then there is a curve $C \subset Y$ along which the $H^0(X_y, K_{X_y})$ are locally constant vector spaces and the $R_{m,y} \subset S^m H^0(X_y, K_{X_y})$ are locally constant subspaces. This implies that the canonical models of the X_y do not vary along C, which contradicts our assumption about injectivity of Kodaira-Spencer maps. □

In general for

$$f : X \to Y$$

where we assume

- X_y is general type for $y \in Y$;
- $\mathrm{Var}\, f = \dim Y$

the canonical bundle $K_y \to X_y$ may not have sections. The assumptions are that for general $y \in Y$ the pluricanonical map

$$\varphi_{mK_{X_y}} : X_y \to \mathbb{P}H^0(X_y, K_{X_y}^{\otimes m})^*$$

is a rational map with image a birational model of X_y, and that as y varies these images modulo projective transformations also vary.

The question is

> *How can one apply Hodge theory to use the $H^0(X_y, K_{X_y}^{\otimes m})$ in the study of the variation of the $\varphi_{mK_{X_y}}(X_y)$'s?*

The idea here dates to Kawamata and has been extensively developed by Vieweg [V1], [V2], [V3], Kollár [K2], Paún [P], Vieweg-Zuo [VZ] and others.

Inverting somewhat the historical development, if one wants positivity, i.e., sections, of $f_*(\omega_{X/Y}^{\otimes m})$ and of $\det f_*(\omega_{X/Y}^{\otimes m})$, then one may try to find positively curved metrics. For a rank r vector bundle

$$E \to M$$

over a compact, complex manifold M it is classical that sections of the bundle that span the fibres give a classifying map from M to the Grassmanian of r-dimensional quotient spaces of $H^0(M, E)$, and one may use the pullback of the metric in the universal quotient bundle to define metrics in $E \to M$ and in the dual bundle.

Suppose however we have sections not of E but of $S^m E \to X$ that span all fibres, then using the metric in $S^m E$ as described above we may define a norm (*not* a metric) in $E \to Y$ by

$$\|e\| = |e^m|^{1/m}$$

where the right-hand side is the m^{th} root of the length given by the metric in the fibres of $S^m E \to M$. This is a *Finisler metric*, not a metric in the usual sense. However it is a natural construction, and through the work of [P], [Z], [LSZ] and others it has been found that many of the properties of usual metrics their curvatures extend to the Finisler case.

An important part of the story seems to be that the Finisler metric in $E \to M$ induces a metric in the fibres of $\mathcal{O}_{\mathbb{P}E^*}(1) \to \mathbb{P}E^*$. Here we are using the convention that for a vector bundle $F \to M$ the fibres of $\mathbb{P}F$ are given by

$$(\mathbb{P}E)_p = \mathbb{P}F_p^*.$$

Then for $\lambda \in F_p^*$ with corresponding point $[\lambda] \in (\mathbb{P}F)$,

$$\mathcal{O}_{\mathbb{P}F}(1)_{(p,[\lambda])} \cong F_p/\lambda^\perp.$$

With this convention, for $\mathbb{P}E \xrightarrow{\pi} M$ we have

$$\pi_* \mathcal{O}_{\mathbb{P}E}(m) = S^m E.$$

Returning to the situation of $f : X \to Y$ what is suggested is that one use in $f_*(\omega_{X/Y}^{\otimes m})$ the norms

$$\|\psi\|^2 = \int_{X_y} (\psi \wedge \overline{\psi})^{1/m} \tag{10}$$

(here ignoring the constant depending only on the dimension of X_y to make the right-hand side positive). The question now is

What can one say about the curvature of (10)*?*

This is where Hodge theory comes in.
To explain this we first give some general remarks. Given a line bundle $L \to M$ and a section $s \in H^0(M, S^m L)$ there is an m-fold cyclic branched covering

$$\widetilde{M}_s \xrightarrow{p} M \tag{11}$$

where $\widetilde{M}_s \subset L$; i.e., in the total space of $L \to M$, $\widetilde{M}_s$ is given by the graph of the multi-valued section $s^{1/m}$.

To be precise we should use the m^{th} power map

$$\begin{tikzcd} L \arrow[r] \arrow[d] & L^m \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd} \tag{12}$$

and take the inverse image under the top map of the usual graph of s. This construction is much studied; we list here the most important properties for our use:

- $\widetilde{M}_s \xrightarrow{p} M$ is branched over the divisor $(s) \subset M$; we denote by $R \subset \widetilde{M}_s$ the ramification locus;
- $\widetilde{M}_s$ is smooth if (s) is; in this case
$$\begin{aligned} K_{\widetilde{M}_s} &= p^* K_M + R \qquad \text{(Riemann-Hurwitz)} \\ &= p^* K_M + (m-1) p^* L; \end{aligned}$$
- if $L \to M$ is ample, then $K_{\widetilde{M}_s}$ is ample for $m \gg 0$.

We now take $L = K_M$ and $s = \psi \in H^0(M, \Omega_M^n)$. Then we have

$$p_* K_{\widetilde{M}_\psi} \cong \bigoplus_{i=1}^{m} K_M^{\otimes i}. \tag{13}$$

In fact, $\widetilde{M}_\psi \xrightarrow{p} M$ is a cyclic covering given by a μ_m-action on $\widetilde{M}_\psi$ and (13) is the eigenspace decomposition of the direct image of $K_{\widetilde{M}_\psi}$ under this action.

- There is an inclusion

$$H^0(M, K_M^{\otimes m}) \hookrightarrow H^0(\widetilde{M}_\psi, K_{\widetilde{M}_\psi}) \subset H^n(\widetilde{M}_\psi, \mathbb{C}) \tag{14}$$

  together with a canonical element

$$\psi^{1/m} \in H^0(\widetilde{M}_\psi, K_{\widetilde{M}_\psi}).$$

- Taking M to be an X_y this gives a norm on the line bundle

$$\mathcal{O}_{\mathbb{P}f_*(\omega_{X/Y}^{\otimes m})^*}(1)$$

defined over an open set in $\mathbb{P}f_*\omega_{X/Y}^{\otimes m} := P_m$. Since

$$H^0(Y, f_*(\omega_{X/Y}^{\otimes m})) \cong H^0(P_m, \mathcal{O}_{\mathcal{P}_m}(1)) \tag{15}$$

one may hope that positivity properties of $\mathcal{O}_{\mathcal{P}_m}(1)$ will lead to sections in the left-hand side of (15), and multiplying these by elements in $H^0(Y, K_Y^m)$ will produce sections in $H^0(X, K_X^m)$.

- The needed positivity of $\mathcal{O}_{P_m}(1)$ results from the positivity of
$$(i/2)\partial\overline{\partial}\log\|\psi\|^2$$
due to the curvature properties of the Hodge bundles and using (14) above.

There is an extensive literature here. We refer to the recent paper [LSZ] for a discussion of the result that is needed here and a guide to further literature.
There are obviously a plethora of technical issues to be dealt with. Some are those generally encountered in birational geometry and use resolution of singularities (to deal with singular phenomena occurring along normal crossing divisors), base change and normalization, etc.
Others are the types of singularities encountered in Hodge theory (cf. [GG] for a general discussion of these).

Even for smooth X_y, although the norm

$$\|\psi\|^2 = \int (\psi \wedge \overline{\psi})^{1/m}, \qquad \psi \in H^0(X_y, K^m_{X_1})$$

is continuous it is not smooth; singularities are encountered when the divisor (ψ) becomes singular. This is to be expected as the above construction lead to a variation of Hodge structure over $\mathbb{P}H^0(X_y, K^m_{X_y})$ and singularities of the VHS arise when the $\widetilde{X}_{y,\psi}$ become singular.
What we have tried to do in this lecture is seek to isolate some of the essential Hodge theoretic aspects of the proof of the Iitaka conjecture under assumptions that essentially sweep the singularity issues under the rug.

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