Example $\vec{R}(t) = (t-1)\vec{i} + (t^2-2t+1)\vec{j}$

① Find equation of tangent line at time $t=2$

② Find the curvature κ

③ Find tangential & normal components of accel.

Idea: Many of the same computations go into figuring out these three things.

① $$\ell(u) = \underbrace{\vec{R}(2)}_{\text{point on line}} + \underbrace{u}_{\text{Parameter}} \underbrace{\vec{R}'(2)}_{\text{vector along line}}$$

(The parameter t is already used in this problem. So we should choose a different variable for this line. I used u.)

② $$\kappa = \left|\frac{d\vec{T}}{ds}\right| \text{ and } \frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\cdot\frac{ds}{dt}$$

$$\frac{ds}{dt} = |\vec{v}| = |\vec{R}'|, \quad \vec{T} = \frac{\vec{v}}{|\vec{v}|}, \quad \vec{v} = \vec{R}'$$

③ $$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|\vec{v}|, \quad a_N = \left(\frac{ds}{dt}\right)^2\kappa = |\vec{v}|^2\kappa$$

So we should first calculate $\vec{v}$, then $|\vec{v}|$, then $\vec{T} = \frac{\vec{v}}{|\vec{v}|}$, $\frac{d\vec{T}}{dt}$, and $\frac{d|\vec{v}|}{dt}$

Calculations

Pay attention + Algebra may help simplify.

$$\vec{R} = \langle t-1,\ t^2-2t+1 \rangle \qquad \vec{R}(2) = \langle 1, 1 \rangle$$

$$\vec{v} = \vec{R}' = \langle 1,\ 2t-2 \rangle = \langle 1,\ 2(t-1) \rangle$$

$$\vec{R}'(2) = \langle 1, 2 \rangle$$

$$\boxed{① \quad \vec{l}(u) = \langle 1, 1 \rangle + u\langle 1, 2 \rangle = \langle 1+u,\ 1+2u \rangle}$$

$$|\vec{v}|^2 = 1^2 + (2(t-1))^2 = 1 + 4(t^2-2t+1) = 4t^2-8t+5$$

$$\frac{ds}{dt} = |\vec{v}| = \sqrt{4t^2-8t+5}$$

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \underbrace{\frac{1}{|\vec{v}|}\langle 1, 2(t-1) \rangle}_{\text{This may be a more convenient form.}} = \frac{\langle 1, 2(t-1) \rangle}{\sqrt{4t^2-8t+5}}$$

Due to Quotient rule, when we calculate $\frac{d\vec{T}}{dt}$ we will need $\frac{d|\vec{v}|}{dt}$. So let's do that first.

$$\frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\sqrt{4t^2-8t+5}\right) = \frac{1}{2}\,\frac{8t-8}{\underbrace{\sqrt{4t^2-8t+5}}_{\text{this denominator is familiar}}} = \underbrace{\frac{4(t-1)}{|\vec{v}|}}_{\text{this is nicer to work with}}$$

$$③ \quad a_T = \frac{d^2s}{dt^2} = \frac{d|\vec{v}|}{dt} = \frac{4t-4}{\sqrt{4t^2-8t+5}}$$

$$= 4(t-1)\left(1+4(t-1)^2\right)^{-1/2}$$

Calculations Continued

$$\frac{d\vec{T}}{dt} = \left\langle \frac{d}{dt}\frac{1}{|\vec{v}|}, \frac{d}{dt}\frac{2(t-1)}{|\vec{v}|} \right\rangle =$$

$$= \left\langle \frac{-\frac{d}{dt}|\vec{v}|}{|\vec{v}|^2}, \frac{2|\vec{v}| - 2(t-1)\frac{d|\vec{v}|}{dt}}{|\vec{v}|^2} \right\rangle$$

$$= \left\langle -\frac{4(t-1)/|\vec{v}|}{|\vec{v}|^2}, 2\frac{|\vec{v}| - (t-1)\,4(t-1)/|\vec{v}|}{|\vec{v}|^2} \right\rangle$$

$$= \left\langle \frac{-4(t-1)}{|\vec{v}|^3}, 2\frac{|\vec{v}|^2 - 4(t-1)^2}{|\vec{v}|^3} \right\rangle$$

$$= \frac{1}{|\vec{v}|^3}\left\langle -4(t-1), 2\left(\underbrace{(1+4(t-1)^2)}_{|\vec{v}|^2} - 4(t-1)^2\right) \right\rangle$$

$$= \frac{1}{|\vec{v}|^3}\langle -4(t-1), 2 \rangle$$

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\cdot\frac{ds}{dt} \quad \text{so} \quad \frac{1}{|\vec{v}|^3}\langle -4(t-1), 2\rangle = \frac{d\vec{T}}{ds}|\vec{v}|$$

$$\frac{d\vec{T}}{ds} = \frac{2}{|\vec{v}|^4}\langle -2(t-1), 1\rangle \qquad K = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|^4}\underbrace{\sqrt{4(t-1)^2+1}}_{=|\vec{v}|} \quad (!!!)$$

Hence $\boxed{② \ K = \frac{1}{|\vec{v}|^3} = (1+4(t-1)^2)^{-3/2}}$

and $\boxed{③ \ a_N = |\vec{v}|^2 K = \frac{1}{|\vec{v}|} = (1+4(t-1)^2)^{-1/2}}$