Ergodic properties of some catalytic particle systems

joint work with M. Kang

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January 2012 / IRS conference at IHP

The underlying process

- $\tilde{\mathbf{X}}_t$, $t \geq 0$ Markov process on open $D \subseteq \mathbb{R}^d$ killed at ∂D $P^D(t, x, dy) = P(\tilde{\mathbf{X}}_t \in dy | \mathbf{X}_0 = x)$ the transition probability functions

  define a Dynkin-Feller semigroup $P_t^D f(x) = \int_D f(y) P^D(t, x, dy)$, $f \in C_0(D)$
- How to continue?

  Restart afresh at a random point x with distribution $\nu(\xi, dx)$ where ξ is the exit point. Continue indefinitely the new process $\mathbf{X}_t$ with transition probabilities $P(t, x, dy)$.
- Denote τ_n the boundary hits and $\lim_{n \to \infty} \tau_n = \tau^*$ possible explosion time
- Catalytic = contact with a set ∂D. Other scenarios.

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Explosion and other questions

- Is it Markovian? Yes, $\xi \to \nu(\xi, dx)$ measurable, etc. ✓
- Does it end in finite time $P(\tau^* < \infty) > 0$? (explode) i.e. the transition kernel is defective $P(t, x, D) < 1$.
  In the diffusive case a hard problem
- Is it Feller?
  Sufficient condition: If $\xi \longrightarrow \nu(\xi, dx) \in M_1(D)$ is continuous
  Example: FV with $N \geq 3$ particles is not
- Is it ergodic? What is the invariant measure?
  When D bounded, $\tilde{\mathbf{X}}_t$ irreducible, the "boundary chain" has compact state space
  Answer: yes, in most cases of interest.
- What is the spectral gap λ ?
  Doeblin theory is satisfactory for existence of $\lambda > 0$.
  Question for FV: $\lambda = \lambda_N \sim O(1)$ as $N \to \infty$?
- Does $\mathbf{X}_t$ give information on $\tilde{\mathbf{X}}_t$?
  The role of the qsd (Ferrari-Maric 2006)

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Example1: BM with rebirth

- $d = 1$, $D = (a, b)$, $a < 0 < b$, $\tilde{\mathbf{X}}_t$ is BM (diffusion, etc) and $\nu(\xi, dx) = \nu(dx)$
  constant redistribution function
  Model related to Barrier options/ math finance
  $\nu = \delta_0$ G-Kang 2001 - explicit computation
  Results on ergodic behavior: do parallel motions (driven by the same $\tilde{\mathbf{X}}_t$) meet in finite time? According to commensurability of the starting points (path collapse) G-Kang 2003, 2007

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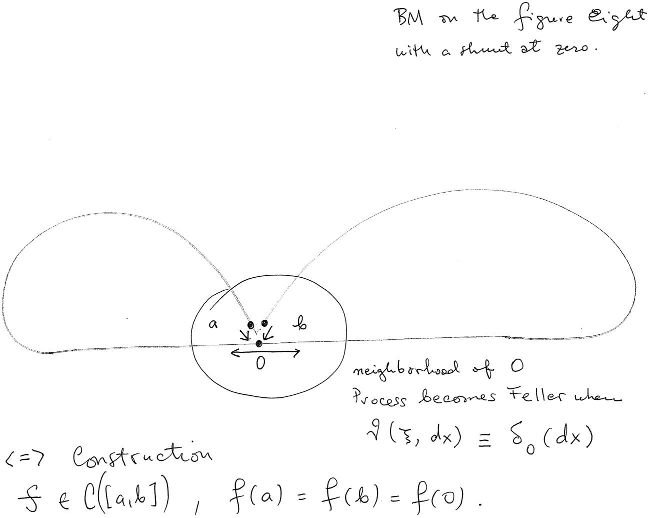

Figure: Shunt on the figure eight.

Example1: BM with rebirth

- $d \geq 2$ G-Kang 2007 analytic semigroups and a proof using Doeblin theory, ergodicity, spectral gap
- ν constant, $\nu(\xi, dx)$ continuous in ξ proof of spectral gap, functional analytic methods BenAri - Pinsky 2005, 2007
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Example 2: A Bak-Sneppen type model

- N particles move in $[0, 1]$ with reflection at 1 and killed at 0. Each particle has a set of *neighbors* (x_i has neighbors x_{i-1} and x_{i+1} but other choices are possible).
  The particle killed, together with its *neighbors* are redistributed iid uniformly (again generalizations are abundant)
- Not mean field for local neighborhood, has strong hierarchical correlations
- Mean field case (when any particle may be chosen as neighbor, uniformly) has hydrodynamic limit when $N \to \infty$ = the normalization of a one particle law with birth instead of killing as in the FV case.

Example 3: Fleming-Viot branching systems

- $G \subseteq \mathbb{R}^q$, N particles, $d = Nq$, $D = G^N$
  $\nu^N(\xi, dx)$ are degenerate measures distributing the particle at ∂G uniformly to the location of one of the remaining $N - 1$ survivors
  Appears in Burdzy-Holyst-March 2000 and before
  Connection to BM with rebirth in Loebus 2009
- One can genaralize: non-uniform distributions appear naturally in establishing large deviations G-2007
- In general (diffusions) the number of boundary hits is regulated by the moving configuration
  Unlike in
  - Moran particle systems (discretization of the FV measure-valued process)
  - discrete D with uniformly bounded Poisson clocks

`Explosion may happen if infinitely many jumps occur in finite time`

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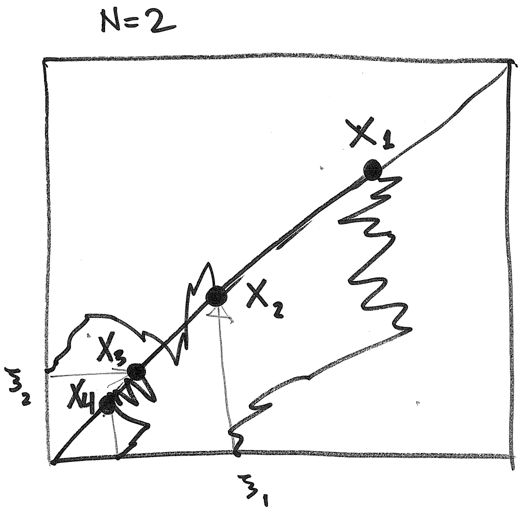
- The interior chain $N = 2$, $D = (0, 1)$



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$N = 2$ the redistribution is continuous, not tight in $M_1(D)$

$\{\nu(\xi, dx)\}_{\xi \in \partial D}$ not tight because at the corner the measure is in $M_1(\bar{D})$ but not in $M_1(D)$.

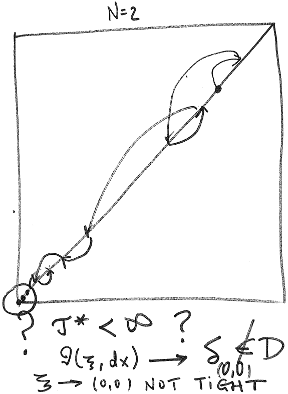

Figure: $\nu(\xi, dx)$ is continuous in ξ.

Example 3: Fleming-Viot branching systems

- $N \geq 3$ not continuous, nor tight in $M_1(D)$ Example $D = (0,1)^3$ arbitrary $b >> \epsilon > 0$

  $\xi' = (0, \epsilon, b)$, $\xi'' = (\epsilon, 0, b)$
  $\nu(\xi', \phi) = \frac{1}{2}(\phi(\epsilon, \epsilon, b) + \phi(b, \epsilon, b))$
  $\nu(\xi'', \phi) = \frac{1}{2}(\phi(\epsilon, \epsilon, b) + \phi(\epsilon, b, b))$
- At the *edge* $\epsilon = 0$ (codimension ≥ 2) the limits are not equal.

The interior set

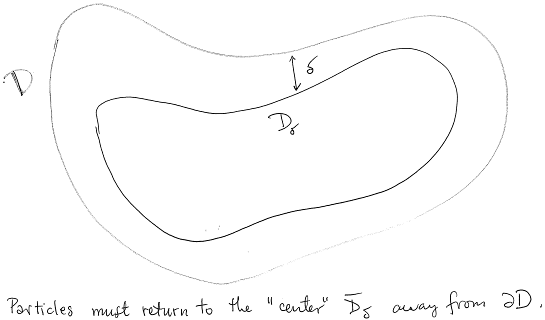

Figure: Interior set D_δ and D.

FV for diffusions/ sufficient conditions for existence

- Assume $E_x[\tau^D] < \infty$
  e.g. D bounded, D positive half line with negative drift
- $D_\delta = \{x \in D | d(x, \partial D) > \delta\}$ interior set (center)
  $\alpha(\delta)$ first hitting time of $\bar{D}_\delta$
  $I(\delta)$ number of jumps until $\alpha(\delta)$, $I(\delta) = J(\alpha(\delta) \wedge \tau^*)$

$$\{I(\delta) < \infty\} = \{\alpha(\delta) < \tau^*\} \ a.s. \qquad \Rightarrow \{\alpha(\delta) < \infty\}$$

Theorem

(Process is non-explosive)
$P_x(I(\delta) < \infty) = 1 \Leftrightarrow P_x(\alpha(\delta) < \tau^*) = 1 \Rightarrow$ *non-explosive*
Note: $I(\delta)$ needs not be uniform in x

Exponential ergodicity/sufficient conditions

Theorem

$\lim_{l\to\infty} \sup_{x\in D\setminus D_\delta} P_x(l(\delta) > l) = 0$ implies
(i) non-explosion (existence of an honest process)
(ii) $\lim_{t\to\infty} \sup_{x\in D\setminus D_\delta} P_x(\alpha(\delta) > t) = 0$ implies the local Doeblin condition

Why?

- The set $F = \bar{D}_\delta$ is attractive and a Doeblin set because $p(T,x,y) \geq p^D(T,x,y) \geq \inf_{x,y\in F} p^D(T,x,y) > 0$

$$p(t,x,y) = p^D(t,x,y) +$$

$$\int_0^t \int_{\partial D} p(t-s,z,y)\nu_\xi(dz) P_x(x(\tau^D -) \in d\xi, \tau^D \in ds)$$

Exponential ergodicity/sufficient conditions

- (C1) $\exists\, m > 0 \qquad \inf_{x \in D \setminus D_\delta} P_x(I(\delta) \le m) \ge c_1 > 0$

  $$\Downarrow$$

  $P_x(I(\delta) < \infty) = 1$ implies the process is nonexplosive
  (C1) true for all except FV
  $m = 1$ in diffusion with rebirth (Example 1)
  $m = N$ in Bak-Sneppen (Example 2)
- (C2) $\{\nu(\xi, dx)\}_{\xi \in \partial D}$ tight implies (C1) with $m = 1$
  Not true for Bak-Sneppen or F-V on "edges"

Interior and boundary chains/ invariant measure

- Interior and boundary chains
  $\lambda(x, d\xi)$ harmonic measure centered at $x \in D$.
- Markov chain on D (interior chain)
  $S(x, dx') = \int_{\partial D} \lambda(x, d\xi)\nu(\xi, dx')$
- Markov chain on ∂D (boundary chain)
  $R(\xi, d\xi') = \int_D \nu(\xi, dx)\lambda(x, d\xi')$
  ∂D compact $\Rightarrow \exists$ invariant probability measure
- Let $\mathcal{L}$ be the infinitesimal generator of the killed process $\tilde{\mathbf{X}}_t$
  $K(x, x')$ Green function for $\mathcal{L}$ with Dirichlet b.c.
  $\mu_X(dx)$ interior invariant measure for the interior chain S
  $\mu(dx) = Z^{-1} \int_D K(x, x')\mu_X(dx')$

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Invariant measure: the FV case

- $\mathbf{X}_t = (x_t^1, x_t^2, \dots, x_t^N)$
  Up to a boundary hit the particles are i.i.d. processes killed at ∂G with transition probabilities $P_x^G(x(t) \in dy)$ with generator L
  Empirical measure process $\mu^N(t, dy) = \frac{1}{N} \sum_{i=1}^N \delta_{x_t^i}(dy)$
  Empirical measure under equilibrium $\mu^N(dy)$
- FV case: not a product measure
- $\mu^N(dx) \Rightarrow m(dx)$ quasi invariant measure

Hydrodynamic limit

- **Hydrodynamic limit** LLN for the trajectories

Theorem (G-Kang 2004)

$\mu^N(0, dy) \Rightarrow \rho_0(x)$ *initial profile*
$\mu^N(t, dy) \Rightarrow \mu(t, dy) = \rho(t, y)dy$
LLN for the empirical measure: the solution is deterministic and solves in weak sense the equation

$$\partial_t \rho = L^* \rho + A'(t)\rho \quad \rho(0, y) = \rho_0(y)$$

$\exp(-A(t)) = P^G_{\rho_0}(x(t) \in G) = P_x(\tau^G > t)$

- Interpretation: $A^N(t) = \frac{1}{N}\{\text{number of jumps up to time } t\}$
  $\lim_{N\to\infty} A^N(t) = A(t)$
- Proof: tightness on the Skorohod space, Ito formula

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Invariant measure: the FV case

- Since

$$\lim_{N\to\infty} \mu^N(t,dy) = \mu(t,dy) = \frac{P^G_{\rho_0}(x(t)\in dy)}{P^G_{\rho_0}(x(t)\in G)}$$

  Conditional on survival up to time $t > 0$
  $P^G_{\rho_0}(x(t)\in G) = P_{\rho_0}(\tau^G > t)$
  Connection with quasi-invariant measures (Ferrari-Maric)

- Under the invariant measure we obtain $0 = L^*\rho + \lambda_1\rho$
  L symmetric (e.g. BM) $A'(t) = \lambda_1 > 0$ the spectral gap
  $\rho = \Phi_1$ first eigenfunction (normalized)

Quasi invariant measures

$x(t)$ Markov process on G; killed at the boundary of G
Determines a Dynkin-Feller semigroup P_t^G with generator L and Green function $K(x,dy) = \int_0^\infty P^G(t,x,dy)$

Theorem

Assume that $E_x[\tau^G] < \infty$ for any $x \in \Lambda$.
(i) If there exists $k > 0$ and a probability measure $m(dx)$ such that $\int m(dx)K(x,dy) = km(dy)$, then $m(dx)$ is a quasi-invariant probability measure and $k = E_m[\tau^G]$.
(ii) If m is a quasi-invariant probability measure for the semigroup and $k = E_m[\tau^G] < \infty$, then $K(x,\cdot)$ is finite for all x m - a.s. and $mR_0 = km$.

Perron-Frobenius and Krein-Rutman theorems

Lemma

Assume that $E_x[\tau^G] < \infty$ *for any* $x \in \Lambda$ *and* ν *is a probability measure. Then* ν *is a left eigenfunction of the Green function* K *corresponding to* $k > 0$ *if and only if* ν *is a left eigenfunction of the infinitesimal generator* L *corresponding to* $-1/k = \lambda_1$*.*

- Results on the eigenfunctions and eigenvalues of a strictly positive operator are available as soon as G is compact.
- Perron-Frobenius theorem (finite dimensional case)
- The infinite dimensional case is covered by the Krein-Rutman theorem.
- To obtain nontrivial results on qsd one needs to look for dynamics with non-compact semigroups. A simple example is motion on the half line with drift towards the origin. (multiple qsd) Ferrari-Martinez-Picco 1992

Commutative diagram

- From $P^G(t, x, dy)$ generate $P^N(t, x, dy)$ the corresponding FV process
-

$$
\begin{array}{ccc}
(FV) = P^N(t,x,dy) & \xrightarrow{t\to\infty} & \mu^N(dy) = (empirical) \\
\Big\downarrow N \to \infty & & \Big\downarrow N \to \infty \\
(Hydrodynamic) = \mu(t,dy) & \xrightarrow{t\to\infty} & m(dy) = (qsd)
\end{array}
$$

Directions:

- Estimates on correlations Asselah-Ferrari-Groisman 2010
- Uniform lower bound (in N) for the spectral gap

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Commutative diagram

- From $P^G(t,x,dy)$ generate $P^N(t,x,dy)$ the corresponding FV process
- $$\begin{array}{ccc} (FV) = P^N(t,x,dy) & \stackrel{t\to\infty}{\longrightarrow} & \mu^N(dy) = (empirical) \\ \Big\downarrow N\to\infty & & \Big\downarrow N\to\infty \\ (Hydrodynamic) = \mu(t,dy) & \stackrel{t\to\infty}{\longrightarrow} & m(dy) = (qsd) \end{array}$$

  Directions:
- Estimates on correlations Asselah-Ferrari-Groisman 2010
- Uniform lower bound (in N) for the spectral gap

FV case - Nonexplosion

- **F-V case:** $\tau^* = +\infty$
- A potential theory argument: Many boundary visits $\Rightarrow$ many particles hang around a set they should not even see (lower dimension) Bienek-Burdzy-Finch 2011

Theorem (G-Kang 2010)

Non-explosion for diffusions with smooth bounded coefficients on domains with quasi-distance to the boundary.

- $G' = G \setminus \bar{G}_\delta \qquad \phi \in C^2(G') \cap C(\overline{G'})$
  $\phi(x) > 0$ on $G' \qquad \phi = 0$ on $\partial G \qquad \inf_{x \in G'} L\phi(x) > -\infty$
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FV case - Nonexplosion

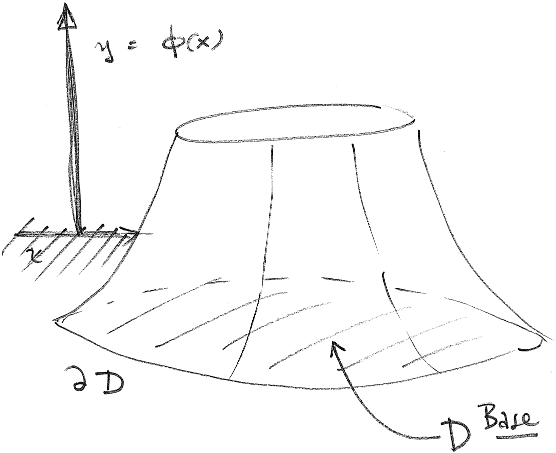
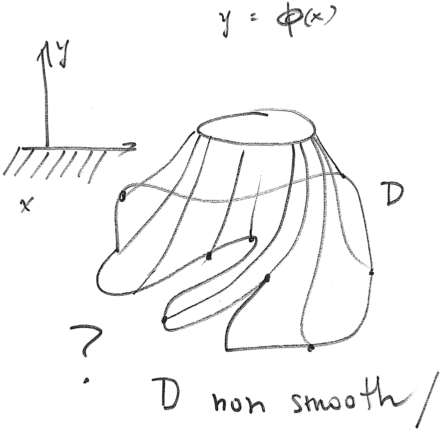


Figure: Interior set D_δ and D.

FV case - Nonexplosion



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Test function ϕ - Eikonal equation/Distance function

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- $\partial G \in C^2$ sufficient condition; $\phi(x) = d(x, \partial G)$ solving the eikonal equation $||\nabla \phi(x)||^2 = 1$.
- G interior sphere condition, Green function $K(\cdot, x') \in C^1$
  $\phi(x) = K(x, x')$, $x' \in G_{2\delta}$
- similar result with first eigenfunction $\Psi(x) \in C^1$
  Hopf's maximum principle
- True for all N and G bounded Lipschitz domain with integrable Martin kernel

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FV case - Nonexplosion

- $D_\delta = \{x \in D | d(x, \partial D) > \delta\}$ interior set (center)
  $\alpha(\delta)$ first hitting time of $\bar{D}_\delta$
  $I(\delta)$ number of jumps before $\alpha(\delta)$
  We need $P_x(I(\delta) < \infty) = 1$ nonexplosive
- $\mathbf{X}_t \in F_k =$ there are exactly k particles in $G \setminus \bar{G}_\delta$
  $I'(\delta)$ number of jumps before reaching $D \setminus F_N$.
- Goal is to enter $\bar{D}_\delta = \bar{G}_\delta^N = \bar{F}_0$

FV case; the "ladder" scheme

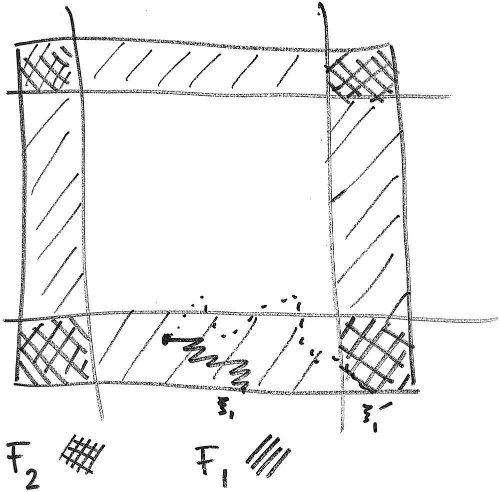
- **Step 1.** We must exit F_N (all are near the boundary). Most work is to ensure that at least one particle eneters the center of the set. $\forall\, x \in F_N \qquad P_x(I'(\delta) < \infty) = 1$

  $I'(\delta)$ number of boundary hits until exiting F_N
- **Step 2.** From F_k to F_{k-1} we use the facts
  - we may always choose the partilce in the interior upon redistribution from the boundary
  - the "interior" particle did not go too far from the center with positive probability

  2.1 $\forall\, x \in F_k \qquad P_x(x(\tau^D-) \in \partial F_k \cap \partial D) \geq c'_k > 0$

  2.2

  $\forall\, \xi = x(\tau^D-) \in \partial F_k \qquad \nu_\xi(x(\tau^D) \in \cup_{j=0}^{k-1} F_j) \geq c''_k > 0$



It is easier to exit through F_1

How to bring at least one perticle to the center

- $G' = G \setminus \bar{G}_\delta \qquad \phi \in C^2(G') \cap C(\overline{G'})$
  $\phi(x) > 0$ on $G' \qquad \phi = 0$ on ∂G
  $||\nabla\phi(x)|| \geq c_- > 0$ on $\bar{G}'$ (can be relaxed very much)
  $y_i(t) = \phi(x_i(t)) \qquad 1 \leq i \leq N$
  $r(t) = (y_1^2(t) + \ldots + y_N^2(t))^{\frac{1}{2}}$

Lemma

If $(\ln r(t))_{t\geq 0}$ *(local) sub-martingale then* $E_x[l'(\delta)] < \infty$.

Proof

1. $L \ln r(t) \geq 0$ *between jumps*

2. $E_x[\ln r(\tau) - \ln r(\tau-)|\mathcal{F}_{\tau-}] \geq U > 0$

$E_x[l'(\delta)] \leq U^{-1}\left[E[\ln r(\alpha'(\delta))] - \ln r(0)\right] < \infty$

The sub-martingale

- **Proof of 2) in the Lemma.** Each boundary hit “costs” a minimum amount in the test function $\ln r(t)$

  $\ln r(\tau) - \ln r(\tau-) \geq \frac{1}{2}\ln\left(1 + \frac{y_j^2(\tau-)}{r^2(\tau-)}\right)$

  for indices j such that $x_j(\tau-) \notin \partial G$

$$E_x[\ln r(\tau) - \ln r(\tau-)] \geq E_x\left[E_{x_i(\tau-)}\left[\frac{1}{2}\ln\frac{r(\tau)}{r(\tau-)}\,\middle|\, x_i(\tau-) \in \partial G\right]\right]$$

$$\geq \frac{1}{2(N-1)}\sum_{j\neq i}\ln\left(1 + \frac{y_j^2(\tau-)}{r^2(\tau-)}\right)$$

$$\geq \frac{1}{2(N-1)}\ln\left(1 + \frac{1}{\sum_{j'\neq i}\left(\frac{y_{j'}(\tau-)}{y_{max}(\tau-)}\right)^2}\right)$$

$$\geq \frac{1}{2(N-1)}\ln\left(1 + \frac{1}{N-1}\right) := U > 0$$

The sub-martingale

- **Proof of 1) in the Lemma.** Similar to a Bessel process
  $\tilde{b}_i(t) = L\phi(x_i(t))\,, \qquad \tilde{\sigma}_i(t) = ||\sigma^*(x_i(t))\nabla\phi(x_i(t))||$
  $dy_i(t) = \tilde{b}_i(t)dt + \tilde{\sigma}(t)d\tilde{w}_i(t)\,, \qquad y_i(0) = \phi(x_{i0})$
  $dr(t) = B(t)dt + S(t)dW(t)$
  $B(t) = \frac{1}{2r(t)}\left(2\langle \mathbf{y}(t), \tilde{b}(t)\rangle + Tr(\tilde{\sigma}(t)\tilde{\sigma}^*(t)) - \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^2(t)}\right)$
  $S(t) = \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||}{r(t)}$
  $\ln r(t)$ sub-martingale if $2r(t)B(t) - S^2(t) \geq 0$
  $\geq N\Big(- c(\phi)\delta + \sigma_0^2(\inf||\nabla\phi(x)||)^2\Big) - 2||\sigma||^2(\sup||\nabla\phi(x)||)^2$

$$N > 2\frac{||\sigma||^2}{\sigma_0^2}\left[\frac{\sup||\nabla\phi(x)||}{\inf||\nabla\phi(x)||}\right]^2$$

Exponential ergodicity by coupling

- **Exponential ergodicity** proof by coupling
  $G' = G \setminus \bar{G}_\delta \qquad \phi \in C^2(G') \cap C(\overline{G'})$
  $\forall x \in G' \quad \phi(x) > 0; \qquad \phi|_{\partial G} = 0; \qquad \phi|_{\partial G_\delta} = 1$
  Proof by coupling $z_i(t)$ follows $y_i(t)$ suppressing jumps
  $dz_i(t) = B(t, \mathbf{x}(t))dt + S(t, \mathbf{x}(t))dw_i(t)$
  $\alpha_y(\delta) \leq \alpha_z(\delta) < \infty$
  $\sup_{\mathbf{x} \in D} E_{\mathbf{x}}[e^{b\alpha_y(\delta)}] < \infty$ with $b > 0$
  $\Downarrow$
  exponential ergodicity

N=2 - explicit formulas

- **Case $N = 2$, $d = 1$, BM with negative drift**
  Interior chain $(X_n)_{n \geq 0}$ $\qquad S(x, dy) = P(X_1 \in dy \,|\, X_0 = x)$

$$S(x, dy) = 2 \int_0^\infty P^G(t, x, dy) P_x(\tau^G \in dt)$$

$$p^G(t, x, y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}} \right) e^{-\mu(y-x) - \frac{1}{2}\mu^2}$$

$$E_x[\tau_1 \wedge \tau_2] = E_x[X^2] \sim o(x) \,, \qquad \lim_{x \to 0} \frac{E_x[X]}{x} = 2 \,. \qquad (1)$$

N=2 - explicit formulas

- **Proposition**
  When $\mu = 0$, the distribution of $V = X_n/X_{n-1}$ is independent of the starting point x having density

$$f_V(v) = \frac{8v}{\pi[(v-1)^2+1][(v+1)^2+1]} \,.$$

$f_V(v) \sim O(v)$ at $v = 0$ and $f_V(v) \sim O(v^{-3})$ at $v = +\infty$
$E[V^a] < \infty$ up to $a < 2$
$\mu_V = 2$, $\sigma_V^2 = \infty$ and $E[\ln V] \approx 0.34$.
(LLN) $\quad \frac{\ln X_n}{n} = \frac{1}{n}\left(\ln x + \sum_k \ln \frac{X_k}{X_{k-1}}\right) \to E[\ln V] > 0$

Immortal particle

- Labeled particle system $(x_i(t), \eta_i(t))$, $\eta_i \in \mathcal{C}$
  When $x_i \to x_j$ then $\eta_i \to \eta_j$
  τ_L first time when there is only one label
  Theorem $P_{\mathbf{x}}(\tau_L < \infty) = 1$
  Proposition All particles alive at time t can be traced continuously to an ancestor from time $t = 0$.
  $\Downarrow$
  Theorem There exists a unique immortal particle.