Mathematical Principles of Natural Philosophy
Sir Isaac Newton

Definition 1
The quantity of matter is the measure of the same, arising from its density and bulk conjointly.
(What is mass?)

Definition 2
The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.
(Displacement arises from momentum [velocity x mass].)

Definition 3
The *vis insita,* or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest…
(Inertia)

Definition 4
An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.
(Definition of force.)

Law I
Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Law II
The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.
(Change of motion [acceleration] is proportional to force. Force is directional.)

Law III
To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to the contrary parts.

About Law II
a=kF
Where "a" is acceleration, "k" is proportionality constant [something to do with mass] and "F" is force.

Rewrite: F=ma (note that k= $\dfrac{1}{m}$)

Rewrite: a= $\dfrac{1}{m}$F   m $\neq$ 0

Meaning of the Letters:
m=mass   assume m is constant
F=net force acting on the body
a= acceleration – change in motion/change in velocity

Velocity is change in position
$\frac{dx}{dt}(t)$ is velocity

Change in velocity is acceleration
$$a(t) = \frac{dv}{dt}(t) = \left(\frac{d}{dt}(\frac{dx}{dt}(t))\right) = \frac{d^2}{dt^2}x(t)$$

Now a= $\frac{1}{m}$F is a "2nd- order" differential equation:

$$\frac{d^2}{dt^2}x(t) = \frac{1}{m}F(x,\frac{dx}{dt},t)$$ ←F can be a function of position, time, velocity, etc.

First, we need an expression for the force. Then we need to solve this differential equation to determine position $x(t)$ on a function of t.

To find $x(t)$, we need to specify initial position $x(o)$ and also initial velocity $\frac{dx}{dt}(0)$.

Mass-spring example (simple harmonic motion)
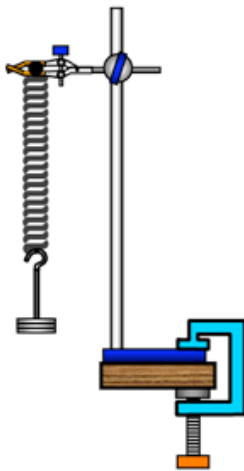
(resourcefulphysics.org)

(We must confine this motion to a straight line)
We must determine a coordinate system.
Assume two forces acting on the mass: gravity and the spring
Zero is at the middle of the hanging mass at rest when the two forces negate each other.
Assume positive direction (+) is downward, so negative direction (-) is upward.

When we pull the mass down, value x will be positive.
Also, we assume there is no friction.

Hooke's Law:
Spring force is proportional to displacement.
$F = -kx$ k is a positive spring constant that must be determined experimentally

Back to Newton:
$$\frac{d^2x}{dt^2} = \frac{1}{m}F$$
$$\frac{d^2x}{dt^2} = \frac{1}{m}(-kx)$$
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

<u>Next</u>: we need to fix *x(0)* and $\frac{dx}{dt}(0)$.

<u>Question</u>: How do you find *x(t)=*?

In the mass-spring, we can write down a formula for *x(t)*.

(hint: $x(t) = \sin\sqrt{\frac{k}{m}}t\ldots$)

Most equations of Newton cannot be solved explicitly by hand because there are no formulas.

Important computational "trick":
To solve a 2nd order differential equation on a computer, we convert a 2nd order equation into a pair of 1st order equations.

Let $x_1(t) = x(t)$ be position and $x_2(t) = \frac{dx}{dt}(t)$ be velocity.

We want differential equations for $x_1(t)$ and $x_2(t)$, so we derive both sides:

$$\frac{dx_1(t)}{dt} = \frac{dx}{dt}(t) \qquad\qquad \text{So, } \frac{dx_1}{dt} = x_2(t)$$

$$\frac{dx_2(t)}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}(t)\right) \qquad\qquad \frac{dx_2(t)}{dt} = ?$$

Everything on the right must be in terms of $x_1(t)$ and $x_2(t)$.

<u>Recall</u>: For mass-spring problem, $\frac{d^2x}{dt^2} = -\frac{k}{m}x$.

So, $\dfrac{dx_1}{dt} = x_2(t)$ and $\dfrac{dx_2}{dt} = -\dfrac{k}{m}x_1(t)$.

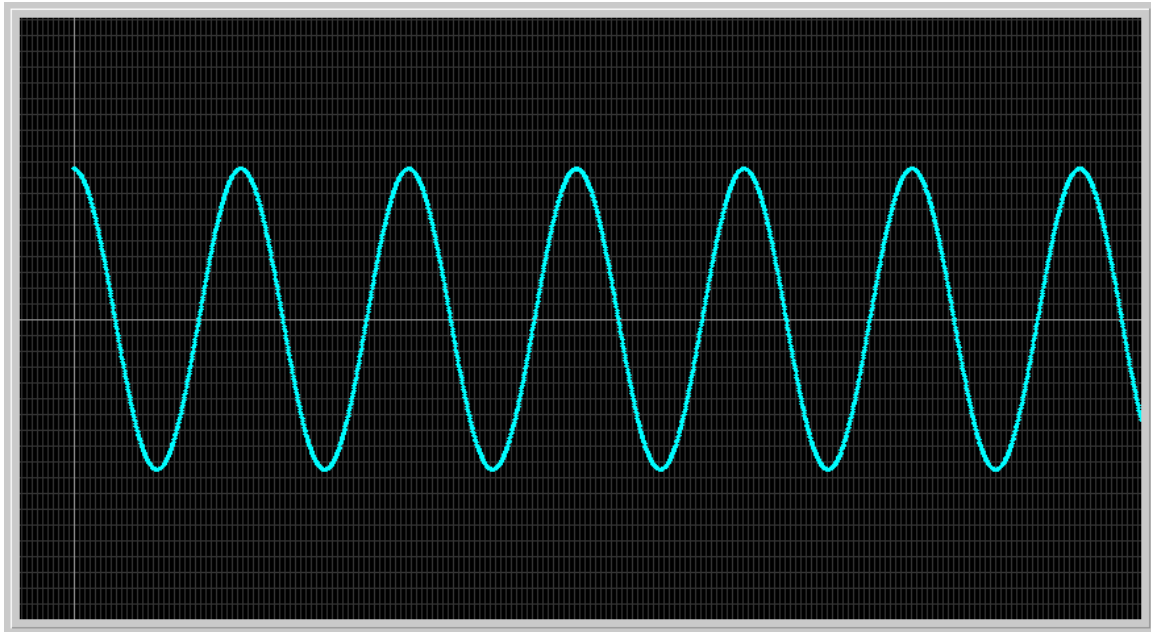In Phaser,
Linear Oscillations ODE
$$x_1 = x_2$$
Equation text:
$$x_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{f}{m}\cos(at)$$
Set c and f equal to zero.

Initial conditions: $x_1=1$, $x_2=0$
Remember x_1 is position and x_2 is velocity, so our mass is pulled down one unit and there is no velocity.

For xi vs time, under current view we plot only position (x_1). Graph point size increased to 4 to make it easier to see.

As time goes on, displacement decreases. The mass is returning to point zero and passing the initial rest position. Then position becomes negative as it passes the rest position and moves upward past it. Since there is no friction, it continues oscillating at period 1 since no energy is lost.

Initial conditions for new set (yellow): $x_1=1$, $x_2=1$
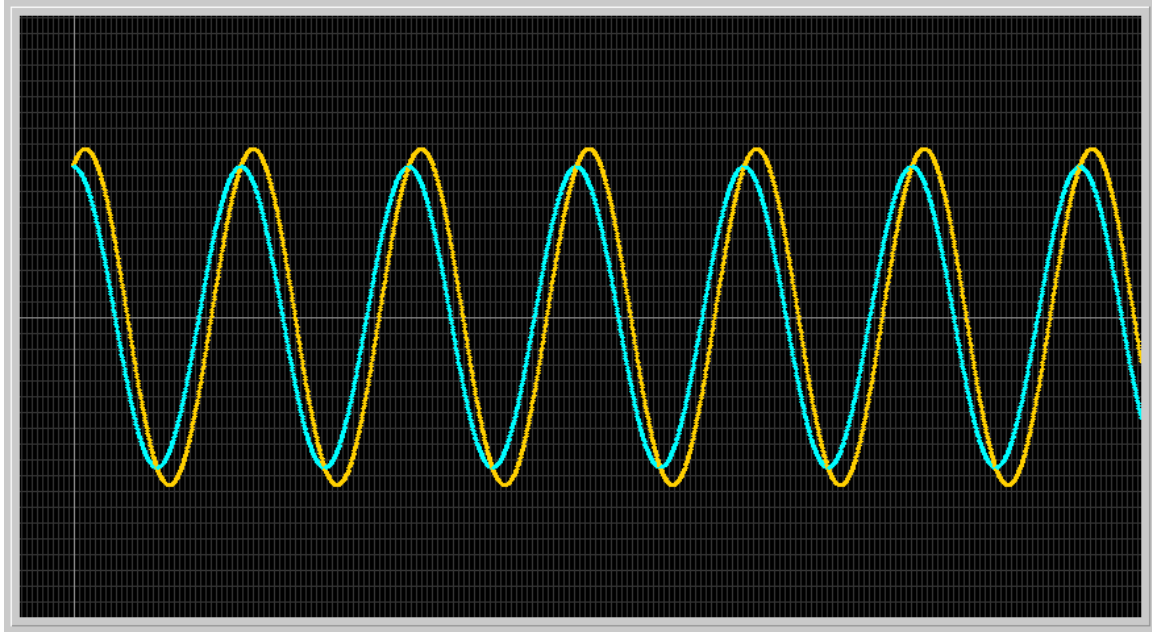
Since there is an initial velocity of 1, the oscillations are different.