

Appeared in **Proceedings of the GARC workshop on GEOMETRY and TOPOLOGY**
‘93 edited by Hong-Jong Kim, Research institute of mathematics, Global analysis research center,
Seoul National University

SU(2) Representation Varieties of 3-manifolds, Gauge Theory Invariants, and Surgery on Knots

Paul Kirk^{*}

*ABSTRACT: Chern-Simons and spectral flow invariants of representations of 3-manifold groups are investigated in the context of Dehn surgery on knots. Various formulae and computational methods are explained, and examples are worked out. The guiding principle is that information about these invariants can be obtained entirely from the representation varieties of the manifolds, i.e. without using any analysis or Differential Geometry. These lectures outline the research investigations of E. Klassen and the author ([KK1]-[KK5]), as well as many others.***

LECTURE 1. *Representation Varieties*

LECTURE 2. *Chern-Simons Invariants*

LECTURE 3. *Spectral Flow*

1 Representation Varieties

1.1 Representation and Character Varieties

Given a compact manifold X , let

$$R(X) = \text{hom}(\pi_1 X, SU(2))$$

^{*} Supported by the DaeWoo foundation, the Global Analysis Research Center at Seoul National University, and the National Science Foundation.

^{**} No serious attempt is made in these notes to be careful and/or consistent with signs and orientation conventions. The reader should refer to the cited articles for precise formulae.

and

$$\chi(X) = \text{hom}(\pi_1 X, SU(2))/\text{conjugation}.$$

These are called the *SU(2)-Representation Variety* and the *SU(2)-Character Variety* of X . They are real-algebraic varieties: if $\pi_1 X$ has the presentation $\pi_1 X = \langle x_1, \dots, x_n | r_1(x_1, \dots, x_n), \dots, r_l(x_1, \dots, x_n) \rangle$, then the relations map

$$r : SU(2)^n \longrightarrow SU(2)^l, \quad r(g_1, \dots, g_n) = (r_1(g_1, \dots, g_n), \dots, r_l(g_1, \dots, g_n))$$

is polynomial, and the map

$$e : R(X) \longrightarrow SU(2)^n, \quad e(\alpha) = (\alpha(x_1), \dots, \alpha(x_n))$$

is an embedding with image $r^{-1}(1, \dots, 1)$. This gives the structure of an algebraic variety (over $\mathbf{R}$ since $SU(2)$ is a variety over $\mathbf{R}$) to $R(X)$ which is independent of the presentation. Then $\chi(X)$ is a variety whose ring of functions is the functions on $R(X)$ which are invariant under the conjugation action $\alpha \mapsto g\alpha g^{-1}$ of $SU(2)$ on $R(X)$.

Notice that R and χ are contravariant functors. Thus a continuous map $f : X \longrightarrow Y$ induces algebraic maps $R(Y) \longrightarrow R(X)$ and $\chi(Y) \longrightarrow \chi(X)$ by restricting representations.

One can make the same definition with any Lie group G replacing $SU(2)$. This complicates matters quite a bit; the resulting representation and character varieties have more singular strata. In any case, $SU(2)$ is complicated enough; it is the “simplest” non-abelian Lie group. Moreover, there are no known examples of 3-manifolds with non-trivial fundamental group but with no non-trivial representations to $SU(2)$. We will therefore stick to $SU(2)$.

1.2 An Important Example

Let $X = T^2$, the torus. Write $\pi_1 T = \mathbf{Z}\mu \oplus \mathbf{Z}\lambda$. Then define a function:

$$\Phi : \mathbf{R}^2 \longrightarrow \chi(T)$$

by sending $(\alpha, \beta) \in \mathbf{R}^2$ to the conjugacy class of representations defined by

$$\mu \mapsto \begin{pmatrix} e^{2\pi i\alpha} & 0 \\ 0 & e^{-2\pi i\alpha} \end{pmatrix}, \quad \lambda \mapsto \begin{pmatrix} e^{2\pi i\beta} & 0 \\ 0 & e^{-2\pi i\beta} \end{pmatrix}.$$

Then $\Phi : \mathbf{R}^2 \longrightarrow \chi(T)$ is a branched cover. In fact, conjugating

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

interchanges the eigenvalues and so $\Phi(\alpha, \beta) = \Phi(\alpha', \beta')$ if and only if

$$(\alpha', \beta') = \pm(\alpha, \beta) + (m, n)$$

for some sign and some $m, n \in \mathbf{Z}$. Thus $(\mathbf{Z}^2 \times | \mathbf{Z}/2)$ acts on $\mathbf{R}^2$ by $(\mu, n, \pm) \cdot (\alpha, \beta) = \pm(\alpha, \beta) + (m, n)$ with quotient $\chi(T)$. The $\mathbf{Z}/2$ comes from the Weyl Group acting on the maximal torus of diagonal matrices. Figure 1 shows a fundamental domain for the action and $\chi(T)$. We call $\chi(T)$ the “pillowcase”.

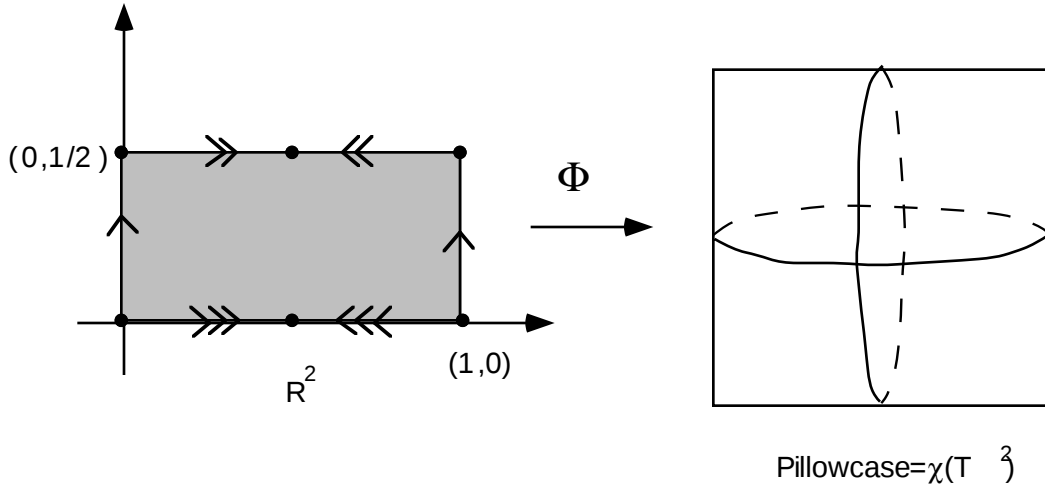


Figure 1.

One caveat about this example: the branched cover $\Phi : \mathbf{R}^2 \rightarrow \chi(X)$ does not induce the correct analytic structure on $\chi(X)$ near the singular points. (This is seen by comparing the dimensions of the Zariski tangent spaces of $\mathbf{R}^2/(\mathbf{Z}^2 \times | \mathbf{Z}/2)$ and $\chi(T)$ at the singular points.) This fact will not be important in these lectures, since we will always work away from the singular points of $\chi(T)$.

Notice that $SU(2)$ is the same as the unit quaternions; the identification is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Leftrightarrow a + bj.$$

For convenience we will use quaternionic notation.

1.3 Some 3-Manifold Topology

We will consider only oriented 3-manifolds in these lectures.

Closed 3-Manifolds decompose into simpler pieces in the following way:

1. *Prime Decomposition.* A 3-manifold has a unique decomposition along 2-spheres as a connected sum $M = M_1 \# M_2 \# \cdots \# M_n$ where each M_i is either *irreducible* (every 2-sphere bounds a ball) or diffeomorphic to $S^2 \times S^1$. [M]
2. *Torus Decomposition.* If M is irreducible and contains an incompressible torus (i.e. a torus T so that the induced map $\pi_1 T \rightarrow \pi_1 M$ is injective), then M has a unique (up to isotopy) maximal collection $T_1 \cup \cdots \cup T_n \subset M$ of embedded tori so that any other incompressible torus is parallel to one of the T_i . Moreover, each piece X_i in the decomposition of M , $M = X_1 \cup \cdots \cup X_k$ obtained by cutting M along the T_i is either Seifert-Fibered or atoroidal. Atoroidal (and a-annular) 3-manifolds are hyperbolic (by Thurston's Uniformization Theorem). [JS], [Jo].
3. If M contains no incompressible torus, but contains an incompressible surface of higher genus, then by Thurston's theorem M is Hyperbolic.

Those 3-manifolds containing an incompressible surface are called *Haken* manifolds, and are relatively well understood by 3-manifold topologists. Less is known about non-Haken manifolds.

We will study the gauge theory invariants of 3-Manifolds with boundary, especially those with torus boundary. These include the important class of *knot complements*.

Definition A *Knot*, K , in a 3-manifold M is a smoothly embedded circle $K : S^1 \hookrightarrow M$. The tubular neighborhood of K is diffeomorphic to $D^2 \times S^1$ and we denote it by $N(K)$. We call the complement in M of an open tubular neighborhood of K the *Exterior* of the knot K , and denote it by X_K . Thus $N(K)$ and X_K are manifolds with boundary a torus.

Dehn Surgery on a knot K in M is a manifold obtained by cutting out a tubular neighborhood $D^2 \times S^1$ of K in M and gluing it back using a self-homeomorphism, h , of the boundary torus. Denote the resulting manifold by $M(K, h)$. Note that $M(K, h) = M(K, h')$ if $h'h^{-1}$ extends over the solid torus $D^2 \times S^1$. Thus the *gluing parameters* lie in

$$\text{Homeo}(S^1 \times S^1) / \text{Homeo}(D^2 \times S^1).$$

Given a knot K in a 3-manifold M , write $M = X_K \cup N(K)$, X_K and $N(K)$ as above. So the boundary torus is $\partial X_K = \partial N(K)$. Let μ be the isotopy class of simple closed curve in the boundary torus which is the boundary of a disc in $N(K) = D^2 \times S^1$. Let λ be an oriented simple closed curve in the boundary torus which intersects μ geometrically (and algebraically) once. (So $\mu \cdot \lambda = 1$.) Then μ, λ gives a coordinate system $S^1 \times S^1$ on ∂X_K .

Note: If M is a homology sphere there is a natural choice for λ determined up to isotopy by the condition that λ generates the kernel of $H_1(\partial X_K) \rightarrow H_1 X_K$. If M is not a homology sphere

there is no canonical choice of λ , and so some arbitrary choice should be made subject to $\mu \cdot \lambda = 1$. None of the results we give in these lectures will depend on the choice of λ .

If we form the manifold $M(K, h)$ by cutting along the torus and gluing back using h , then there exists a relatively prime pair of integers p, q so that the curve $p\mu + q\lambda$ is null homologous in $N(K)$. More precisely, think of $h : \partial N(K) \rightarrow \partial X_K$ and so $h(\mu) = p\mu + q\lambda$ homologically. A simple exercise shows that the ratio $\frac{p}{q} \in \mathbf{Q} \cup \infty$ determines $M(K, h)$ up to diffeomorphism and so we write $M(K, \frac{p}{q})$ for $M(K, h)$. Then $M(K, \frac{p}{q})$ is called “ p over q surgery on K ”. If K is a knot in a homology 3-sphere, then $H_1(M(K, \frac{p}{q})) = \mathbf{Z}/p$.

We will call μ the *meridian* and λ the *longitude*.

The surgered manifold $M(K, \frac{p}{q})$ contains a knot, namely the core $0 \times S^1 \subset D^2 \times S^1 = N(K)$. However, the meridian in $M(K, \frac{p}{q})$ does not equal the meridian in M ; in fact the meridian in $M(K, \frac{p}{q})$ is $\bar{\mu} = p\mu + q\lambda$, where μ and λ refers to the meridian and longitude for M . We can take as longitude $\bar{\lambda} = r\mu + s\lambda$ where $ps - qr = 1$. (Remark: we will often use additive notation when working in $\pi_1 T^2 = H_1 T^2$.)

1.4 Gluing Representations

We use the following notation, if $Y \subset X$ is a subspace and $\rho_X : \pi_1 X \rightarrow SU(2)$ is a representation, then let $\rho_{X|Y} : \pi_1 Y \rightarrow SU(2)$ denote the restriction, i.e. the image under the induced map $R(X) \rightarrow R(Y)$. Similarly, if $[\rho_X] \in \chi(X)$, let $[\rho_{X|Y}]$ denote its restriction to Y using the induced map $\chi(X) \rightarrow \chi(Y)$. Here $[\cdot]$ means conjugacy class.

Suppose that $M^3 = X \cup_\Sigma Y$ is a decomposition of a 3-manifold along a closed surface Σ . Let $[\rho_X] \in \chi(X)$, and $[\rho_Y] \in \chi(Y)$. Then Van Kampen’s theorem says that ρ_X and ρ_Y glue together to give a representation of M if and only if $[\rho_{X|\Sigma}] = [\rho_{Y|\Sigma}]$.

More precisely, let $[\rho_X] \in \chi(X)$, $[\rho_Y] \in \chi(Y)$ and choose representatives $\rho_X \in R(X)$, $\rho_Y \in R(Y)$. If $[\rho_{X|\Sigma}] = [\rho_{Y|\Sigma}]$, then there exists a $g \in SU(2)$ so that

$$\rho_{X|\Sigma} = g\rho_{Y|\Sigma}g^{-1}.$$

Now

$$\pi_1 M = \frac{\pi_1 X * \pi_1 Y}{N(i_X(s)i_Y(s)^{-1} | s \in \pi_1 \Sigma)},$$

and therefore $\rho_X * g\rho_Y g^{-1}$ defines a representation $\rho_M \in R(M)$. The conjugacy class of $\rho_X * g\rho_Y g^{-1}$ may in general depend on the choice of representatives of ρ_X and ρ_Y chosen and the choice of g .

To understand this, we first consider centralizers of subgroups of $SU(2)$. The center of $SU(2)$

is $\{\pm Id\}$. If $H \subset SU(2)$ is a subgroup, then the centralizer of H in $SU(2)$ is

$$Z(H) = \begin{cases} \{\pm Id\} & \text{if } H \text{ is non-abelian,} \\ S_H^1 & \text{if } H \text{ is abelian, but } H \not\subset \{\pm Id\}, \\ SU(2) & \text{if } H \subset \{\pm Id\}. \end{cases}$$

Here S_H^1 denotes the unique maximal abelian subgroup containing H , which is a circle.

Suppose that $[\rho_X] \in \chi(X)$, $[\rho_Y] \in \chi(Y)$, and $[\rho_{X|\Sigma}] = [\rho_{Y|\Sigma}]$ in $\chi(\Sigma)$. Choose representatives ρ_X and ρ_Y so that $\rho_{X|\Sigma} = \rho_{Y|\Sigma}$. Then if $g \in Z(\text{Image } \rho_{X|\Sigma})$, $\rho_{X|\Sigma} = g\rho_{Y|\Sigma}g^{-1}$. Hence $g \mapsto [\rho_X * (g\rho_Yg^{-1})]$ defines a function $Z(\text{Image } \rho_{X|\Sigma}) \rightarrow \chi(X)$. This function surjects to the fiber

$$F([\rho_X], [\rho_Y]) = \{[\rho_M] \mid [\rho_{M|X}] = [\rho_X], [\rho_{M|Y}] = [\rho_Y]\}.$$

Now if $g, h \in Z(\text{Image } \rho_{X|\Sigma})$, then $[\rho_X * (g\rho_Yg^{-1})] = [\rho_X * (h\rho_Yh^{-1})]$ if and only if there exists an $l \in Z(\text{Image } \rho_X)$ so that $g^{-1}lh \in Z(\text{Image } \rho_Y)$. So, for example, if ρ_X and ρ_Y are non-abelian this happens only if $g = \pm h$, and so in this case $F([\rho_X], [\rho_Y]) \cong Z(\text{Image } \rho_{X|\Sigma}) / \pm Id$.

In the particular case when Σ is a torus T^2 , then $\pi_1 T$ is abelian, hence $Z(\text{Image } \rho_{X|T})$ is either a circle or $SU(2)$ depending on whether the image of $\rho_{X|T}$ is non-central or central.

Consider the special case of surgery on a knot K in a 3-manifold M . As before we decompose $M = X_K \cup N(K)$ and let $\mu, \lambda \in \pi_1 \partial X_K$ be the meridian and longitude. In this case the fundamental group of M is just a quotient of the fundamental group of X_K obtained by killing the meridian:

$$\pi_1 M = \pi_1 X_K / N(\mu).$$

Similarly for p/q Dehn surgery we have

$$\pi_1(M(K, \frac{p}{q})) = \pi_1 X_K / N(\mu^p \lambda^q).$$

Hence $\rho_X : \pi_1 X_K \rightarrow SU(2)$ extends to $\pi_1 M$ if and only if $\rho_X(\mu) = 1$. Similarly $\rho_X : \pi_1 X_K \rightarrow SU(2)$ extends to $\pi_1(M(K, \frac{p}{q}))$ if and only if $\rho_X(\bar{\mu}) = 1$, where $\bar{\mu} = \mu^p \lambda^q$.

This can easily be understood in terms of the pillowcase: the subvariety

$$\{[\rho] \in \chi(T) \mid \rho(\mu^p \lambda^q) = 1\}$$

is just the image of the line $p\alpha + q\beta = 0$ under the branched cover $\Phi : \mathbf{R}^2 \rightarrow \chi(T)$ of example 1.2. Figure 2 shows the case $\frac{p}{q} = \frac{3}{5}$.

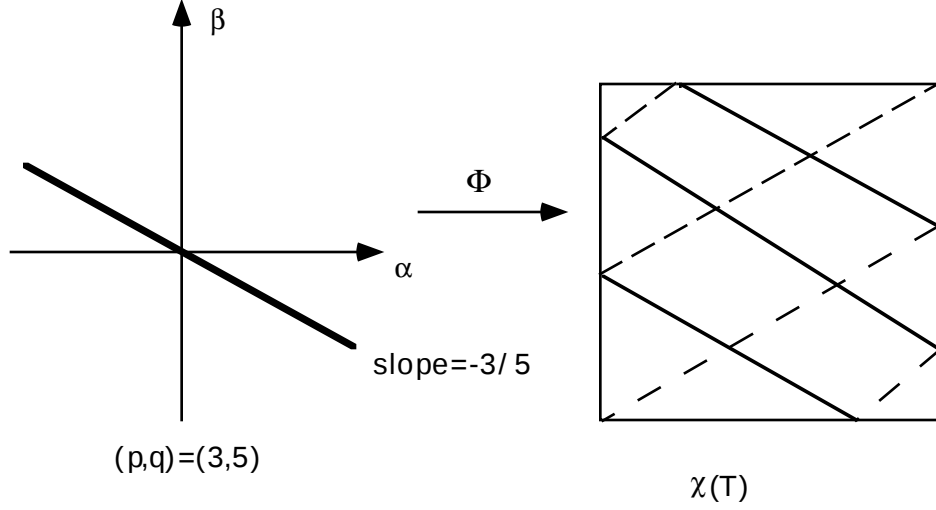


Figure 2.

It turns out that understanding $\chi(X_K)$ and the restriction $\chi(X_K) \rightarrow \chi(T)$ is the key to computing Chern-Simons and spectral flow invariants of 3-manifolds. There are a few papers which calculate the character varieties of knot complements and other 3-manifolds; See [K1], [K2], [KK1], [Bo], [B], [Fr], and [H].

Before we list a few examples, we collect a few facts which hold in general for knots in a homology sphere.

1. If $K \subset M$ is a knot in a homology sphere, then $H_1 X_K = \mathbf{Z}$. Therefore, the representations of $\pi_1 X_K$ with abelian image are independent of K . Now $\chi(\mathbf{Z}) = SU(2)/conjugation \cong [-2, 2]$, the homeomorphism given by taking the trace. Also, μ generates $H_1 X_K$ and $\lambda = 0$ in $H_1 X_K$. Thus writing

$$\chi(X) = \chi^a(X) \cup \chi^*(X)$$

where $\chi^a(X)$ denotes the classes of abelian representations, we see that $\chi^a(X)$ is homeomorphic to an interval and the restriction $\chi^a(X) \rightarrow \chi(T)$ has image the bottom horizontal edge of the pillowcase. Notice that $\chi^a(X)$ is parameterized by the representations $\mu \mapsto e^{it}$, $t \in [0, \pi]$ and the endpoints of the interval correspond to the trivial representation and the non-trivial central representation. See Figure 3. The reader should use this picture to count the number of points in $\chi^a(M(K, \frac{p}{q}))$.

In general $H_1 X_K$ depends only on the homology class of K in M , and so $\chi^a(X_K)$ also depends only on the homology class of K in M . To obtain more interesting information about the knot K one must study the non-abelian representations $\chi^*(X_K)$.

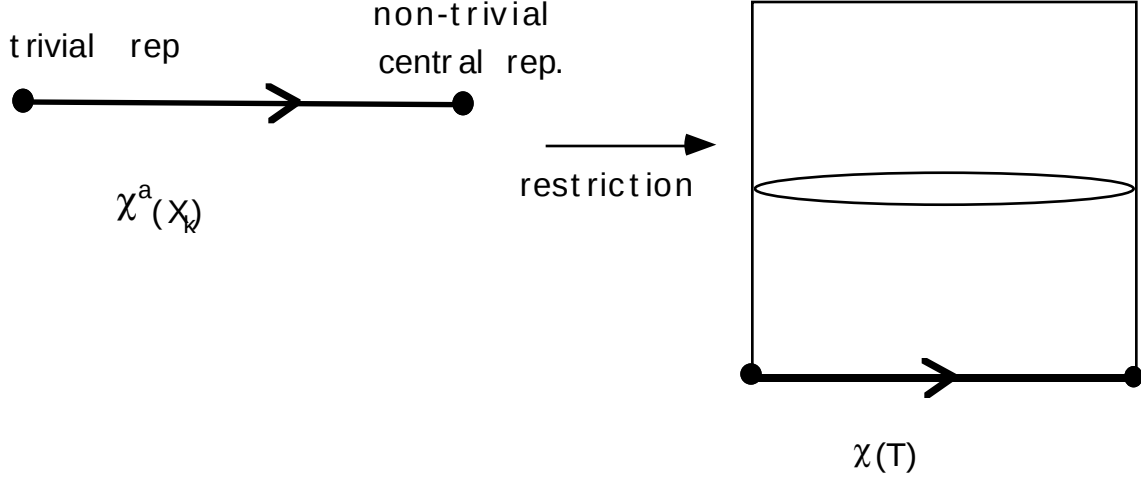


Figure 3.

2. The image of $\chi(X_K) \rightarrow \chi(T)$ is a 1-dimensional subvariety (in fact a Lagrangian subvariety of the symplectic variety $\chi(T)$, as we will see later). To explain this we first review the relevant cohomology ideas:

Let X be a space with $\pi_1 X = \langle x_1, \dots, x_n | r_1, \dots, r_l \rangle$. The *Zariski Tangent Space* of the variety $R(X)$ at a representation $\rho : \pi_1 X \rightarrow SU(2)$ is isomorphic to $\ker dr_{e(\rho)}$, where $r : SU(2)^n \rightarrow SU(2)^l$ is the relations map $(g_1, \dots, g_n) \mapsto (r_1(g_1, \dots, g_n), \dots)$, and $e : R(X) \subset SU(2)^n$ is the embedding $\rho \mapsto (\rho(x_1), \dots)$ as explained above.

Let $su(2)$ denote the Lie algebra of $SU(2)$, and $ad\rho : \pi_1 X \rightarrow GL(su(2))$ the composite of ρ with the adjoint representation of $SU(2)$. We identify the Zariski tangent space with the group cohomology of $\pi_1 X$ with coefficients in $su(2)$: the p -cocycles are functions $(\pi_1 X)^p \rightarrow su(2)$, the differential $d : C^0(\pi_1 X; ad\rho) \rightarrow C^1(\pi_1 X; ad\rho)$ takes $v \in su(2) = C^0(\pi_1 X; ad\rho)$ to $x \mapsto v - ad\rho(x) \cdot v$, and the differential $d : C^1(\pi_1 X; ad\rho) \rightarrow C^2(\pi_1 X; ad\rho)$ takes $c \in C^1(\pi_1 X; ad\rho)$ to $(x, y) \mapsto c(x) + ad\rho(x) \cdot c(y) - c(xy)$. Thus the 1-cocycles are crossed homomorphisms, and the 1-coboundaries are principal homomorphisms.

Any element of $\ker dr_{e(\rho)}$ can be written in the form

$$(v_1 \rho(x_1), \dots, v_n \rho(x_n))$$

for some $(v_1, \dots, v_n) \in su(2)^n$. Then it is not hard to see that the assignment $x_i \mapsto v_i$ defines a 1-cocycle $v \in Z^1(\pi_1 X; ad\rho)$. Conversely, if v is a 1-cocycle then $(v(x_1) \rho(x_1), \dots, v(x_n) \rho(x_n)) \in \ker dr_{e(\rho)}$. This identifies the tangent space of $R(X)$ at ρ with $Z^1(\pi_1 X; ad\rho)$. Finally, a tangent vector is tangent to the orbit of the conjugation action of $SU(2)$ on $R(X)$ if and only if the corresponding cocycle is a coboundary. Therefore,

$$T_\rho \chi(X) \cong H^1(X; ad\rho).$$

Now suppose that $\partial X = \Sigma$. The differential of the restriction $\chi(X) \rightarrow \chi(T)$ is just the induced map on cohomology $H^1(X; \text{ad}\rho) \rightarrow H^1(\Sigma; \text{ad}\rho_\Sigma)$. If X is a 3-manifold then Poincaré duality implies that this image is middle dimensional.

Now an easy computation shows that

$$T_\rho \chi(T) = \begin{cases} \mathbf{R}^2 & \text{if } \rho \text{ is not central,} \\ \mathbf{R}^6 & \text{if } \rho \text{ is central.} \end{cases}$$

The pillowcase is a smooth 2-dimensional variety except at the 4 corners corresponding to the central representations. Thus if $\partial X = T^2$ the image of $\chi(X) \rightarrow \chi(T)$ is generically 1-dimensional.

The computation of the Zariski tangent space at the corners of the pillowcase also show that our branched cover $\Phi : \mathbf{R}^2 \rightarrow \chi(T)$ is not analytic at the corners. Indeed, the Zariski tangent space of $\mathbf{R}^2 / \pm 1$ at 0 is 3 dimensional, not 6 dimensional.

3. For a knot in S^3 , the meridian μ normally generates $\pi_1 X_K$. Therefore, if $\rho \in R(X_K)$ sends the meridian to $\pm Id$, then ρ must be central. Thus there are only 2 representations of $\pi_1 X_K$ which map to a corner of the pillowcase, namely the trivial representation and the non-trivial central representation. In particular, this implies that the restriction of every non-abelian representation of $\pi_1 X_K$ to the pillowcase misses the corners. This is not true for knots in general 3-manifolds, or even in homology spheres.

4. Another restriction on the image $\chi(X_K) \rightarrow \chi(T)$ is that the image must be a *Legendrian* subvariety [He]. See section 2.9 below.

5. If the dimension of $\chi(X_K)$ is larger than 1, then X_K contains a closed incompressible surface. See [K1] for a proof. Usually, if X_K contains a separating incompressible torus which is not boundary parallel, then the dimension of $\chi(X_K)$ is greater than 1; one sees this by using the gluing construction described above to “bend” a representation along the separating torus

There are not too many more general facts which one can say about representation varieties of knot complements. See [FK] for a theorem on deforming abelian representations of knot groups into non-abelian representations.

1.5 Examples

We give a few examples for knots in S^3 . The arguments can be found in the citations given above.

1. *The Unknot.* U has $\pi_1 X_U = \mathbf{Z}$, and so $\chi(X_K) = \chi^a(X_K)$ which is an interval, as explained above. The image of this interval in the pillowcase is the bottom edge of the pillowcase, i.e. the image of the α -axis under the map $\Phi : \mathbf{R}^2 \rightarrow \chi(T)$. Surgeries on the unknot yields lens spaces; in

fact $M(U, \frac{p}{q}) = L(p, q)$. Thus one can count the points in $\chi(L(p, q))$ by counting the intersections of $\Phi(\beta = 0)$ and $\Phi(p\alpha + q\beta = 0)$ in the pillowcase; there are $[\frac{p-1}{2}]$ such representations.

2. *The Trefoil knot.* Let K denote the trefoil. Then $\chi^*(X_K)$ consists of an open arc, whose endpoints are abelian representations. Its image in $\chi(T^2)$ coincides with the image of the arc

$$\{(t, -6t + \frac{1}{2}) \mid \frac{1}{12} < t < \frac{5}{12}\} \subset \mathbf{R}^2$$

under $\Phi : \mathbf{R}^2 \rightarrow \chi(T)$. It starts and ends at the bottom of the pillowcase and winds twice around.

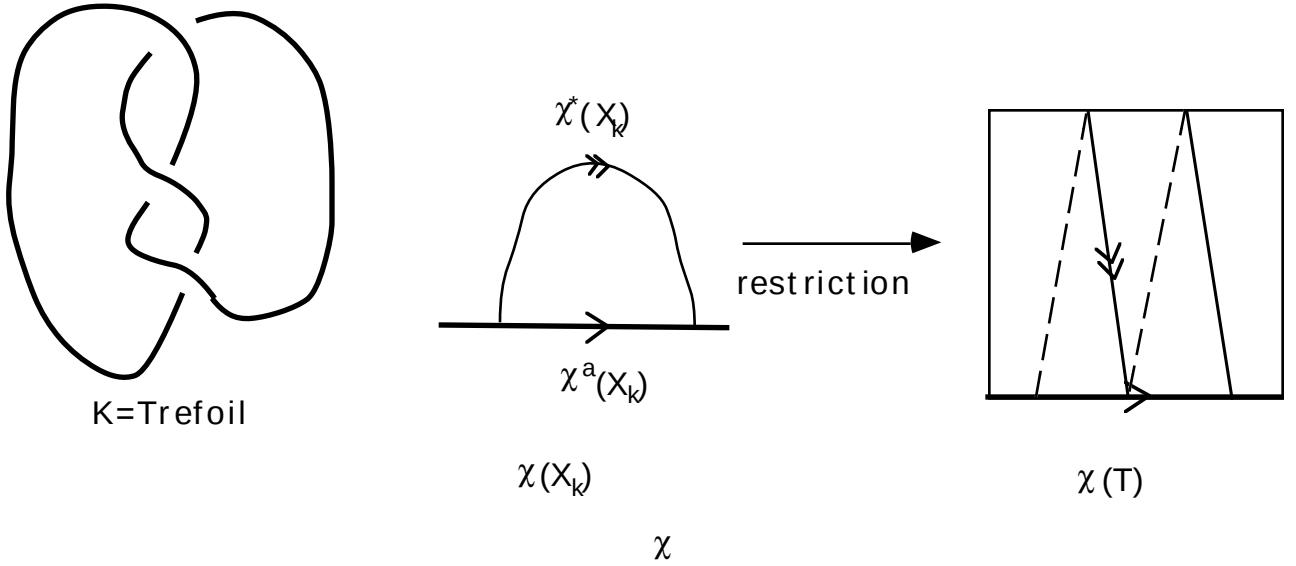


Figure 4.

Arbitrary torus knots can be treated in the same way. One uses the presentation $\pi = \langle x, y \mid x^p = y^q \rangle$ for the (p, q) torus knot and $\mu = x^m y^n$, $\lambda = x^p (x^m y^n)^{-pq}$ where $pn - qm = 1$ to find the image in the pillowcase. With a bit of work one shows that each component of $\chi(X_K)$ is an open arc limiting on the abelian representations; in fact there are $(p-1)(q-1)/2$ such arcs ([K1]).

Any non-abelian representation must take $x^p = y^q$ to ± 1 since this element is central in π . This implies that in any component of $\chi^*(X_K)$, $\rho(\lambda) = \pm \rho(\mu)^{-pq}$. Thus each arc in $\chi(X_K)$ maps into a line segments of slope $-\frac{1}{pq}$ starting and ending along the bottom edge (by “line of slope m ” we mean the image of a line of slope m in $\mathbf{R}^2$ under Φ).

3. *The Figure 8 Knot.* If K denotes the Figure 8 knot then $\chi^*(X_K)$ is a smooth circle. Its image in the pillowcase wraps twice around. For an argument see [K1] or [B].

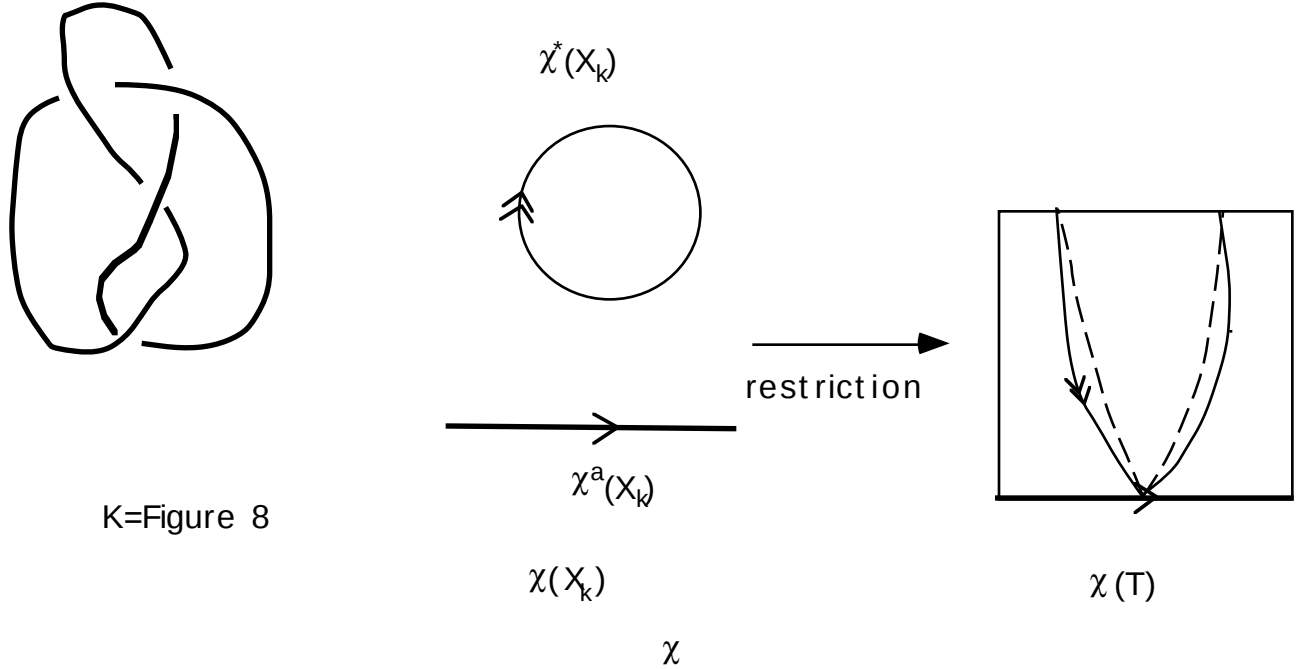


Figure 5.

4. *Two-Bridge Knots.* See [B] for a parameterization of $\chi(X_K)$ for K any two-bridge knot. Burde shows that $\chi^*(X_K)$ is 1-dimensional, consisting of smooth circles and open arcs limiting on the abelian representations. See also [H].

There are some other papers which compute $\chi(X_K)$ for various K and in some cases their image in the pillowcase. [K2] shows how to understand the character varieties of twisted Whitehead doubles of a knot K in terms of $\chi(X_K)$ and $\chi(W)$ where W denotes the Whitehead link complement. The knot polynomial of [CCGLP] cuts out the image $\chi_C(X_K) \rightarrow \chi_C(T)$, where χ_C refers to the $SL(2, \mathbf{C})$ character varieties. The real points of these complex varieties are the union of $SL(2, \mathbf{R})$ and $SU(2)$ character varieties, and one can sometimes use this polynomial and a computer to graph the image of $\chi(X_K)$ in the pillowcase.

1.6 Representations of Dehn Surgery on a Knot

We give some examples which illustrate how to understand the representations of $M(K, \frac{p}{q})$. As we explained, this corresponds to the intersections of the image of $\chi(X_K)$ in the pillowcase with the line segment $\Phi(p\alpha + q\beta = 0)$. Figure 6 shows 3 examples, $-\frac{3}{5}$ surgery on the Trefoil, $\frac{1}{3}$ surgery on the Trefoil, and $\frac{1}{4}$ surgery on the Figure 8.

M=-3/5 surgery on Trefoil

M=1/3 surgery on Trefoil

M=1/4 surgery on Figure 8

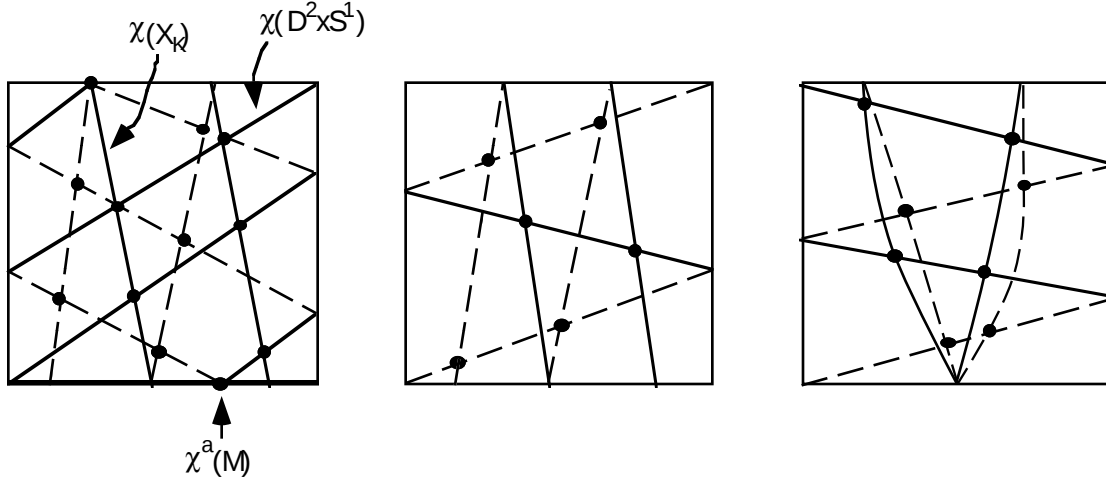


Figure 6.

Using the figure and the considerations of section 1.4, one see that $-\frac{3}{5}$ surgery on the Trefoil has 11 (conjugacy classes of) non-abelian representations, one non-trivial abelian representation, and the trivial representation.

Similarly $\frac{1}{3}$ surgery on the Trefoil has 6 non-abelian (conjugacy classes of) representations, and only one abelian representation, namely the trivial representation. (Remark: Casson's invariant is $3 = \frac{6}{2}$.)

Finally we see that $\frac{1}{4}$ surgery on the Figure 8 has 8 conjugacy classes of non-abelian representations, and only the trivial abelian representation. (Remark: Casson's invariant is $4 = \frac{8}{2}$.)

These pictures will be used to calculate the Chern-Simons and spectral flow invariants of surgeries on knots.

From a theoretical point of view, there is nothing special about knot complements, or even manifolds with toral boundary. However, the torus distinguishes itself from the higher genus surface because $\chi(T)$ is 2-dimensional and has only 2 strata: the abelian non-central representations and the central representations. Higher genus surfaces have, in addition, a strata of non-abelian representations which is $6g - 6$ dimensional. So the torus is the only 2-manifold whose character variety I can draw. From the point of view of 3-manifold topology decompositions along tori have already proven useful. It makes sense to separate the problem of describing $\chi(X)$ into two cases, X a Seifert fibered or X hyperbolic. The Seifert-fibered case is well-understood; see [A], [Bo], and [KK1] among others. Not too much is written about $SU(2)$ representations of hyperbolic manifolds, but $SU(2)$ is a subgroup of $SL(2, \mathbf{C})$ and the literature is replete with papers on $SL(2, \mathbf{C})$ representations of hyperbolic manifolds.

Thinking as a knot theorist, one can view the image of $\chi(X_K)$ in the pillowcase as a knot invariant. It contains a lot of information, for example it can tell whether a knot has Property P, or whether surgery yields a non-cyclic fundamental group. We will see what it tells us about gauge theory invariants in the next lectures. In any case, describing $\chi(X)$ and its image in $\chi(T)$ is a non-trivial and interesting problem which does not involve any analysis or geometry.

2 Chern-Simons Invariants of 3-Manifolds and Decompositions Along Tori.

2.1 Connections

Let $P \rightarrow X$ be a principal $SU(2)$ bundle over a manifold X ; we assume that X has dimension 2 or 3. Since $\pi_i SU(2) = 0$ for $i = 1$ and 2, P is trivializable. For convenience fix a trivialization $P \cong X \times SU(2)$. There are many definitions of connections on P . Pick your favorite definition and let $\mathcal{A}_P$ denote the space of all connections on P . Then:

1. If $A \in \mathcal{A}$ and $r : SU(2) \rightarrow GL(V)$ is any representation, then A defines a covariant derivative

$$d_A : \Omega^p(E) \rightarrow \Omega^{p+1}(E)$$

where E is the associated vector bundle $E = P \times_r V$, and $\Omega^p(E)$ denotes the differential p -forms on X with values in the bundle E . Of course E is itself trivial, since P is, and so $\Omega^p(E) \cong \Omega^p \otimes V$. The covariant derivative satisfies the Leibnitz rule

$$d_A(a \wedge b) = d_A a \wedge b + (-1)^{|a|} a \wedge d_A b.$$

2. The trivial bundle has a distinguished connection, namely the product connection, which we denote by Θ and call the product connection. Its associated covariant derivative is just the usual exterior derivative with vector values which we denote by d :

$$d : \Omega^p \otimes V \rightarrow \Omega^{p+1} \otimes V.$$

3. If $A, B \in \mathcal{A}$, then the difference $d_A - d_B$ is a 0^{th} order operator, i.e. $(d_A - d_B)(f\alpha) = f(d_A - d_B)(\alpha)$ for $f \in C^\infty(X)$, and $\alpha \in \Omega^p(E)$. Thus $d_A - d_B \in \Omega^1(Hom(E, E))$. Using the given trivialization and taking the standard representation $SU(2) \rightarrow GL(\mathbf{C}^2)$ a simple computation shows that $d_A - d_B \in \Omega^1 \otimes su(2)$. Moreover, the map $\mathcal{A} \rightarrow \Omega^1 \otimes su(2)$ taking A to $d_A - d$ is an isomorphism. Thus a trivialization of P induces an identification of $\mathcal{A}$ with $su(2)$ -valued 1-forms. Given a connection A (which we think of as a 1-form) and a representation

$r : SU(2) \longrightarrow GL(V)$ the induced covariant derivative

$$d_A : \Omega^P \otimes V \longrightarrow \Omega^{p+1} \otimes V$$

is given by the formula

$$d_A \alpha = (d + r_*(A))\alpha$$

where $r_* : su(2) \longrightarrow gl(V)$ is the derivative of r , acting on V . We will abuse notation whenever it is convenient and use A to denote both 1-forms and connections, with the understanding that it is only a 1-form with respect to the fixed trivialization.

4. Different trivializations are related by the *gauge group*, also called the *group of gauge transformations*. It is the group

$$\mathcal{G} = C^\infty(X, SU(2)) = Aut(P)$$

of automorphisms of the bundle. Precomposing the fixed trivialization $SU(2) \times X \cong P$ with $g : X \longrightarrow SU(2)$ gives a new trivialization. If $A \in \Omega^1 \otimes su(2)$ denotes the connection 1-form in our fixed trivialization, then in the new trivialization A is replaced by $gAg^{-1} - dg g^{-1}$. (Notice that $dg g^{-1} \in \Omega^1 \otimes su(2)$.) This action of $\mathcal{G}$ on $\mathcal{A}$ corresponds on the level of covariant derivatives to $d_{g \cdot A} \alpha = g(d_A(g^{-1}(\alpha)))$, where g acts on $E = X \times V$ via r .

The orbit space $\mathcal{A} / \mathcal{G}$ of *gauge-equivalence classes of connections* is denoted by $\mathcal{B}$. In a certain technical sense $\mathcal{B}$ is (or more precisely can be completed to be) an infinite dimensional singular Banach manifold.

2.2 Holonomy and Curvature

Given a connection $A \in \mathcal{A}$ consider the linear map $d_A d_A : \Omega^p \otimes \mathbf{C}^2 \longrightarrow \Omega^{p+1} \otimes \mathbf{C}^2$. Using the fact that $d_A(f\omega) = df \wedge \omega + f d_A \omega$ for $f \in C^\infty(X)$, $\omega \in \Omega^p \otimes \mathbf{C}^2$ one computes

$$d_A d_A f \omega = f d_A d_A \omega.$$

Thus $d_A d_A \in \Omega^2(\text{hom}(\mathbf{C}^2, \mathbf{C}^2))$. It is called the *curvature* of A and denoted by $F(A)$ and is easily seen to lie in $\Omega^2 \otimes su(2)$. In terms of our trivialization the curvature of the connection 1-form A is

$$F(A) = dA + \frac{1}{2}[A, A].$$

Given a loop $\gamma : I \longrightarrow X$, then $\gamma^*(A)$ is a connection on $I \times SU(2)$. The ODE:

$$\frac{dg}{dt} = \gamma^*(A)g, \quad g(0) = Id$$

for $g : I \rightarrow SU(2)$ has a solution, and $g(1)$ is called the *Holonomy of A around the loop γ* , denoted by $hol_A(\gamma)$. Thus $A \in \mathcal{A}$ defines a function

$$hol_A : Loops(X, x_0) \rightarrow SU(2).$$

This function factors through $\pi_1(X, x_0)$ (i.e. depends only on the homotopy class of γ) if and only if $F(A) = 0$, i.e. if A is *Flat*.

Let $\mathcal{F} \subset \mathcal{A}$ denote the flat connections. Then the holonomy induces a map

$$hol : \mathcal{F} \rightarrow R(X).$$

The gauge group $\mathcal{G}$ leaves $\mathcal{F}$ invariant, and the holonomy induces a *homeomorphism*

$$hol : \mathcal{F} / \mathcal{G} \cong \chi(X).$$

This is the basic relationship between the algebraic investigations of the first lecture and the analytic objects we are studying now.

2.3 Chern-Simons Invariants

Let $\mathbf{A}$ be a connection on a principal $SU(2)$ bundle Q over a *closed 4-manifold M* . Chern-Weil Theory implies that the integral

$$\frac{1}{8\pi^2} \int_M tr(F(\mathbf{A}) \wedge F(\mathbf{A})),$$

although a priori a real number depending on $\mathbf{A}$, is in fact equal to the integer $c_2(Q)[M]$ and in particular is independent of $\mathbf{A}$.

This immediately suggests defining an invariant for connections on 3-manifolds with values in $\mathbf{R}/\mathbf{Z}$: if A is a connection on a principal $SU(2)$ bundle P over a closed 3-manifold X , let X bound the 4-manifold M , extend the bundle P to Q and A to $\mathbf{A}$. Then define

$$cs_X(A) = \frac{1}{8\pi^2} \int_M tr(F(\mathbf{A}) \wedge F(\mathbf{A})) \in \mathbf{R}/\mathbf{Z}.$$

This gives us a function, the *Chern-Simons Invariant*

$$cs_X : \mathcal{B} \rightarrow \mathbf{R}/\mathbf{Z}.$$

One can also give a direct definition in terms of the connection 1-forms: if $A \in \mathcal{A}$ then using Stokes' theorem one can show:

$$(2.1) \quad cs_X(A) = \frac{1}{8\pi^2} \int_X tr(dA \wedge A + \frac{2}{3} A \wedge A \wedge A).$$

This function has many nice properties and has been the subject of much investigation recently. Taubes ([T]) introduced the idea of viewing cs_X as a Morse function, and by (carefully) mimicking Morse theory, Floer ([F]) has used it to define the “Instanton Homology” groups of X . The critical points of cs_X are the flat connections $\mathcal{F}/\mathcal{G} = \chi(X)$. These instanton homology groups provide a “relative” theory for Donaldson’s 4-manifold theory [DK]. Some of this will be outlined in the next lecture.

One nice use for the restriction of cs to $\chi(X)$ is the proof of Fintushel and Stern that the collection $\{\Sigma(2, 3, 6r - 1)\}$ of Seifert-fibered homology spheres are linearly independent in the homology cobordism group Θ_H^3 . [FS]

A recent development is the “topological quantum field theory” which Witten [W] “defines” using the functional integral

$$Z_k(X) = \int_{\mathcal{A}} e^{2\pi i k \, cs}$$

for any $k \in \mathbf{Z}$. Although this expression does not make mathematical sense, Witten outlines two methods to interpret this integral. The first leads to an Axiomatic definition of 3-manifold invariants which have been rigorously constructed by Reshetikin-Turaev [RT], Walker [Wa], and others. The alternative interpretation is to pretend that the stationary phase expansion of integrals like this in $\mathbf{R}^n$ works for this integral over $\mathcal{A}$. This method leads to an asymptotic expansion (as $k \rightarrow \infty$) of $Z_k(X)$ whose leading term is a sum over the critical points of cs , i.e. over the points in $\chi(X)$, of expressions involving the value of the Chern-Simons invariant at A , the spectral flow of the Hessian of the Chern-Simons invariant from A to Θ , and the Ray-Singer(= Reidemeister) torsion of the chain complex with local coefficients *ad hol* _{A} . One can ask whether these two interpretations are consistent, and we will see an example in the next lecture.

2.4 Surgery and the Chern-Simons Invariants of Flat Connections

Let us now consider cs_M as a function on $\chi(M)$ for a closed 3-manifold M by restricting cs_M to $\chi(M) \cong \mathcal{F}/\mathcal{G} \subset \mathcal{B}$. It has several nice properties.

First, cs_M is an *oriented flat cobordism invariant*. In other words, suppose that $\rho_0 \in \chi(M_0)$, $\rho_1 \in \chi(M_1)$ and there exists a 4-manifold Y such that $\partial Y = X_1 \amalg -X_0$ and a $\rho \in \chi(Y)$ so that $\rho|_{M_0} = \rho_0$, and $\rho|_{X_1} = \rho_1$. Then $cs_{M_0}(\rho_0) = cs_{M_1}(\rho_1)$. The reason for this is that the difference of Chern-Simons invariants is the integral of $tr(F(\mathbf{A}) \wedge F(\mathbf{A}))$ where $\mathbf{A}$ is a flat connection on Y with holonomy ρ . But $\mathbf{A}$ is flat, so $F(\mathbf{A}) = 0$.

A fancy way to say this is that cs defines a homomorphism

$$H_3(BSU(2)^\delta) \rightarrow \mathbf{R}/\mathbf{Z},$$

where $SU(2)^\delta$ means $SU(2)$ with the discrete topology, so $BSU(2)^\delta = K(SU(2), 1)$, the CW complex with fundamental group equal to $SU(2)$ and all other homotopy groups equal to 0. One conjecture which has been around is that the image of this homomorphism lies in the rationals, i.e. in $\mathbf{Q}/\mathbf{Z}$, so that the Chern-Simons invariant of a flat connection on a closed 3-manifold should be rational. Many computations have been made, for example Seifert-Fibered spaces and torus bundles over S^1 , [KK2], [A], and these have rational Chern-Simons invariants.

Another property of cs_M is that it is locally constant, i.e. $cs_M(\rho)$ depends only on the path component of $\chi(M)$ containing ρ . To see this, suppose that $\rho_t, t \in [0, 1]$ is a path of representations of M , and A_t a corresponding path of flat connections so that $hol_{A_t} = \rho_t$. We can view the path A_t as a single connection $\mathbf{A}$ on the 4-manifold $M \times [0, 1]$; then

$$F(\mathbf{A}) = d\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}] = \frac{\partial \mathbf{A}}{\partial t} \wedge dt.$$

Thus $F(\mathbf{A}) \wedge F(\mathbf{A}) = 0$, and so

$$cs_M(\rho_1) - cs_M(\rho_0) = \frac{1}{8\pi^2} \int_{M \times I} tr(F(\mathbf{A}) \wedge F(\mathbf{A})) = 0.$$

This leads to the following idea. Suppose $M^3 = X \cup_\Sigma Y$ is a decomposition of a closed 3-manifold along a surface, and suppose $\rho_0, \rho_1 \in \chi(M)$. We know that if these representations lie on the same path component of $\chi(M)$ then their Chern-Simons invariants are the same, but what about if the restrictions to X or Y (or both) lie in the same path component? In other words, can we obtain a formula for the difference $cs_M(\rho_1) - cs_M(\rho_0)$ if $\rho_0|_X$ and $\rho_1|_X$ lie in the same path component of $\chi(X)$ and/or $\rho_0|_Y$ and $\rho_1|_Y$ lie in the same path component of $\chi(Y)$?

The easiest case to consider is Dehn Surgery. In [KK2] we proved the following theorem.

2.5 Theorem. *Let $K \subset M$ be a knot in a 3-manifold. Let $\rho_0, \rho_1 \in \chi(M)$ and suppose there exists a path $\rho_t : I \rightarrow R(X_K)$ from $\rho_0|_X$ to $\rho_1|_X$. Let $(\alpha(t), \beta(t)), t \in I$ be a path in $\mathbf{R}^2$ such that*

$$\rho_t(\mu) = e^{2\pi i \alpha(t)}, \quad \rho_t(\lambda) = e^{2\pi i \beta(t)}$$

(in other words the restriction of ρ_t to the pillowcase is $\Phi(\alpha(t), \beta(t))$).

Then:

$$cs_M(\rho_1) - cs_M(\rho_0) = -2 \int_0^1 \beta \alpha' dt \in \mathbf{R}/\mathbf{Z}.$$

What this means is that if we know the image of the path $\rho_t : \pi_1 X_K \rightarrow SU(2)$ in the pillowcase then we can compute the difference in Chern-Simons invariants. Thus the difference is determined

by this image. This theorem illustrates the basic philosophy which says that a Gauge theory invariant (namely the difference of Chern-Simons invariants) can be computed entirely from the algebraic data of the map $\chi(X_K) \rightarrow \chi(T)$ for Dehn surgery on a knot.

The fundamental principle which underlies the proof is that one can find a path of connections on M which are flat on X_K and non-flat (excepts at the endpoints) on $N(K) \cong D^2 \times S^1$. Thus the easy part of the connection, i.e. the flat part, lives on the complicated part of the 3-manifold, the exterior of the knot. However, the difficult part of the connection (the non-flat part) lives on an easy space, $D^2 \times S^1$.

2.6 Let us try a computation to illustrate this theorem. Let $M = \frac{1}{2}$ surgery on the trefoil. The following figure shows that $\chi(M)$ has 4 (conjugacy classes of) representations:

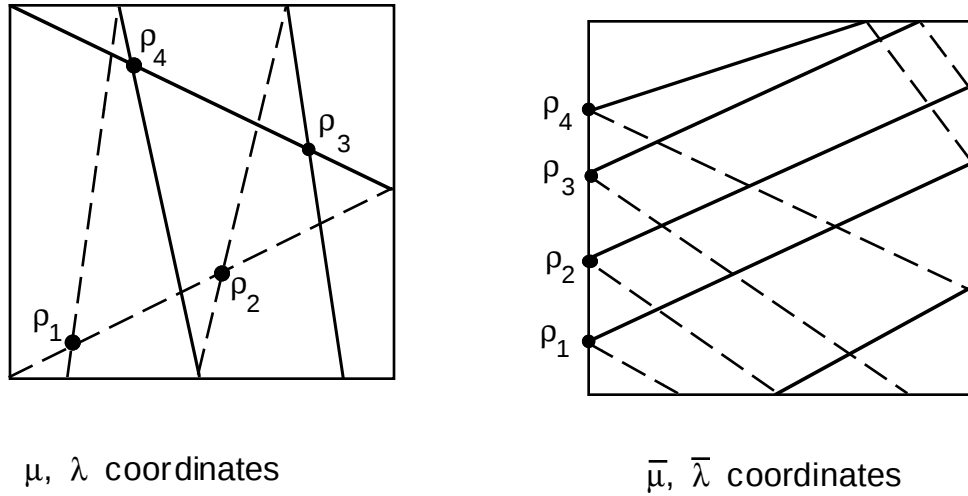


Figure 7.

The picture on the left is given in the coordinates μ, λ of $K \subset S^3$. To apply the Theorem, we need to use a meridian and longitude for $M = S^3(K, \frac{1}{2})$. In particular we need a meridian which bounds a disc in M . This is achieved by using the coordinates $\bar{\mu}, \bar{\lambda}$ defined by:

$$\bar{\mu} = \mu + 2\lambda, \quad \bar{\lambda} = \lambda.$$

Thus the right side of Figure 7 is the image of $\chi(X_K)$ in these new coordinates. This just corresponds to using the new coordinates $\bar{\alpha}, \bar{\beta}$ for $\mathbf{R}^2$, where

$$\bar{\alpha} = \alpha + 2\beta, \quad \bar{\beta} = \beta.$$

We learned in the first section that the image of $\chi^*(X_K) \rightarrow \chi(T)$ (in the α, β coordinates) is the arc $\Phi(t, -6t + \frac{1}{2})$, $\frac{1}{12} < t < \frac{5}{12}$. In the new coordinates we therefore have

$$\bar{\alpha}(t) = t + 2(-6t + \frac{1}{2}) = -11t + 1$$

and

$$\bar{\beta}(t) = -6t + \frac{1}{2}.$$

This gives the arc of slope $\frac{6}{11}$ in the right side of Figure 7. Since the $\rho_i, i = 1, 2, 3, 4$ correspond to the representations of $M = S^3(K, \frac{1}{2})$, they must send $\bar{\mu}$ to Id . This is seen in the right side of Figure 7 by the fact that the ρ_i are aligned on the left edge of the pillowcase, which corresponds to $\rho \in \chi(T)$ such that $\rho(\bar{\mu}) = Id$. Thus the ρ_i correspond to $\bar{\alpha}(t) \in \mathbf{Z}$, and this happens when $t = \frac{1}{11}$ (for ρ_1), $t = \frac{2}{11}$ (for ρ_2), $t = \frac{3}{11}$ (for ρ_3), $t = \frac{4}{11}$ (for ρ_4). So for example,

$$\begin{aligned} cs(\rho_3) - cs(\rho_2) &= -2 \int_{2/11}^{3/11} \bar{\beta}(t) \bar{\alpha}'(t) dt \\ &= -2 \int_{2/11}^{3/11} (-6t + 1/2)(-11) dt = -30/11 \equiv \frac{8}{11} \pmod{\mathbf{Z}} \end{aligned}$$

Some other computations possible using this theorem include an easy proof that the set of Chern-Simons invariants of the lens space $L(p, q)$ is

$$\left\{ -\frac{n^2 r}{p} \mid n = 0, 1, \dots, \left[\frac{p}{2} \right] \right\}$$

where $r \in \mathbf{Z}$ satisfies $qr \equiv -1 \pmod{p}$. The proof is to apply the theorem to surgeries on the unknot. We remark that the collection of Chern-Simons invariants distinguishes homotopy inequivalent lens spaces, although it cannot distinguish homotopy equivalent non-homeomorphic lens spaces (since cs_X is a homotopy invariant).

Letting K be a regular fiber in a Seifert Fibered homology sphere, one quickly reproves Fintushel and Stern's [FS] computation of the Chern-Simons invariants of these 3-manifolds. The argument is basically identical to the computation for $\frac{1}{2}$ surgery on the Trefoil carried out above.

A slightly trickier computation is the Chern-Simons invariant of surgeries on the figure 8 knot. Although one can compute the difference between the Chern-Simons invariants of irreducible representations in the same way, there is no path from an irreducible representation of the figure 8 knot exterior to the trivial representation since $\chi(X_K)$ is not connected (see example 1.5, Figure 5). This is overcome by passing to the complex character variety, i.e. by considering $\text{hom}(\pi_1 X_K, SL(2, \mathbf{C}))$ which turns out to be connected in this case. As a sample application one can compute the Chern-Simons invariants of surgeries on the Figure 8 knot numerically (i.e. by computer). Applying Fintushel and Stern's argument one can then show that $-\frac{1}{3}$ and $-\frac{1}{4}$ surgery on the Figure 8 knot are each linearly independent of the set $\{\Sigma(p, q, pqk - 1)\}$ in the homology cobordism group Θ_H^3 except possibly that $-\frac{1}{4}$ might be homology cobordant to $n\Sigma(2, 3, 5)$ for some $n \geq 0$. For the proofs of the preceding facts see [KK2].

2.7 The S^1 bundle over $\chi(\Sigma)$

What about general decompositions along tori? In [KKR] we needed a computation of Chern-Simons invariants for graph manifolds obtained by gluing together the complements of regular fibers in Seifert Fibered homology spheres in order to compute their Instanton Homology.

What is needed is a definition of Chern-Simons invariants for manifolds with boundary. At first sight this seems problematic since the integral

$$\frac{1}{8\pi^2} \int_X \text{tr}(dA \wedge A + \frac{2}{3} A \wedge A \wedge A) \in \mathbf{R}/\mathbf{Z}$$

is not gauge invariant on a manifold with boundary; that is, it can change by a non-integer if $A \in \mathcal{A}$ is replaced by $g \cdot A$. Thus one cannot use this integral to define an $\mathbf{R}/\mathbf{Z} = S^1$ -valued function on $\chi(X)$.

However, there is a simple solution to this problem which I first saw in [RSW], which shows cs_X is a cross section of a non-trivial S^1 bundle over $\chi(\partial X)$. The bundle is constructed as follows.

Let Σ be a closed oriented surface and let $Q \rightarrow \Sigma$ be a principal $SU(2)$ bundle over Σ . Then define $\theta : \mathcal{A}_Q \times \mathcal{G}_Q \rightarrow S^1$ by

$$\theta(A, g) = \exp(2\pi i(cs(\tilde{A}) - cs(\tilde{g} \cdot \tilde{A})))$$

where $\tilde{A}$ is some extension of A to a connection over a 3-manifold X with boundary Σ , and $\tilde{g}$ is an extension of g to the corresponding bundle over X . The fact that cs_M is well-defined in $\mathbf{R}/\mathbf{Z}$ if M is closed implies that $\theta(A, g)$ is independent of the choice of extensions.

Thus θ defines an action of $\mathcal{G}_Q$ on $\mathcal{A}_Q \times S^1$ covering the $\mathcal{G}_Q$ action on $\mathcal{A}_Q$ by the formula

$$g \cdot (A, z) = (g \cdot A, \theta(A, g)z).$$

One checks that if $g \cdot A = A$, then $\theta(A, g) = Id$, and so the $\mathcal{G}_Q$ -equivariant bundle $\mathcal{A}_Q \times S^1 \rightarrow \mathcal{A}_Q$ has a quotient S^1 -bundle

$$L_\Sigma \rightarrow \mathcal{B}_\Sigma.$$

(Note that Q is necessarily trivial and so L_Σ depends only on Σ .)

It is now a tautology that if $\partial X = \Sigma$, then $c_X = e^{2\pi i cs_X}$ is a cross section of $L_\Sigma \rightarrow \mathcal{B}_\Sigma$ over the restriction $\mathcal{B}_X \rightarrow \mathcal{B}_\Sigma$.

$$\begin{array}{ccc} & & L_\Sigma \\ & \nearrow & \downarrow \\ \mathcal{B}_X & \longrightarrow & \mathcal{B}_\Sigma \end{array}$$

We can restrict to character varieties since $\chi = \mathcal{F}/\mathcal{G} \subset \mathcal{B}$, thereby obtaining a bundle $L_\Sigma \rightarrow \chi(\Sigma)$ such that for any 3-manifold X with boundary Σ , $c_X = e^{2\pi i c_S X}$ defines a lift of the restriction map $\chi(X) \rightarrow \chi(\Sigma)$ as indicated in the next diagram.

$$\begin{array}{ccc} & & L_\Sigma \\ & \nearrow & \downarrow \\ \chi_X & \longrightarrow & \chi_\Sigma \end{array}$$

This line bundle has extra structure. First of all, $\chi(\Sigma)$ has a symplectic structure induced by the cup product on cohomology. To see this, recall that $T_\rho \chi(\Sigma) \cong H^1(\Sigma; \text{ad} \rho)$. The cup product

$$\cdot : H^1(\Sigma; \text{ad} \rho) \times H^1(\Sigma; \text{ad} \rho) \rightarrow H^2(\Sigma, \mathbf{R}) \cong \mathbf{R}$$

induced by the non-degenerate form $(a, b) \mapsto \text{tr}(ab)$ on $\mathfrak{su}(2)$ is skew symmetric and non-degenerate, and induces a symplectic structure on $\chi(\Sigma)$. (Strictly speaking, since $\chi(\Sigma)$ is not a manifold, we should think of $\chi(\Sigma)$ either as a stratified object or just restrict to the top stratum.) In [RSW] a connection on the bundle L_Σ is constructed whose curvature is this symplectic form. Moreover they show that if X is a 3-manifold with boundary Σ the image $\chi(X) \rightarrow \chi(\Sigma)$ is Lagrangian.

In [KK3] we construct the bundle L_Σ directly when Σ is a torus. From our construction the connection on L_T and the symplectic structure on (the top stratum of) $\chi(T)$ is obvious. Moreover, Theorem 2.5 above can be succinctly stated by saying that the lift c_X of $\chi(X) \rightarrow \chi(T)$ to L_T is parallel. We describe this now.

The construction is similar to the one given above, but with a smaller group. Let G be the semi-direct product of $\mathbf{Z}$ and $\mathbf{Z}/2$, acting on $\mathbf{R}^2$ by $(m, n, \pm) \cdot (\alpha, \beta) = \pm(\alpha, \beta) + (m, n)$. We saw in example 1.2 that $\mathbf{R}^2/G$ is a good model for $\chi(X)$. Now extend the action of G to $\mathbf{R}^2 \times S^1$ by the formula

$$(m, n, \pm) \cdot (\alpha, \beta, z) = (\pm\alpha + m, \pm\beta + n, ze^{2\pi i(m\beta - n\alpha)}).$$

Then the trivial bundle descends to give a quotient bundle $\mathcal{L} \rightarrow \chi(T)$.

Moreover, the inner product $\langle \cdot, \cdot \rangle : S^1 \times S^1 \rightarrow S^1$ given by $\langle z, w \rangle = zw$ induces a (fiberwise) inner product $\mathcal{L}_T \times \mathcal{L}_T \rightarrow S^1$. Then [KK3]:

2.8 Theorem.

1. $\mathcal{L}_T = L_T$, and so if X is a 3-manifold with boundary a torus, $c_X = \exp(2\pi i c_S)$ defines a lift of the restriction $\chi(X) \rightarrow \chi(T)$ to $\mathcal{L}_T$.
2. Let $M = X \cup_T Y$ be a closed 3-manifold obtained by cutting M along a torus. If $\rho_M \in \chi(M)$, and $\rho_X = \rho_M|_X \in \chi(X)$, $\rho_Y = \rho_M|_Y \in \chi(Y)$, then

$$e^{2\pi i c_{SM}(\rho_M)} = \langle c_X(\rho_X), c_Y(\rho_Y) \rangle.$$

3. The euler class of $\mathcal{L}_T$ is -1 .

(Remark: It seems that several people knew this type of fact before we did. Moreover it is not hard to prove once one has seen the idea of [RSW].)

2.9 The Connection and Symplectic Form on L_T .

The previous theorem says that if we can compute c_X for manifolds with boundary a torus, part 2 of this theorem together with the gluing results of section 1.4 shows how to compute Chern-Simons invariants of manifolds glued along tori. A more general result holds for manifolds decomposed along a union of tori.

To take advantage of this, we need a theorem like 2.5 to compute c_X for manifolds with toral boundary.

We use the following notation for points in L_T : since L_T is a quotient of $\mathbf{R}^2 \times S^1$ we write $[\alpha, \beta; z]$ for equivalence classes, so for example

$$[\alpha, \beta; z] = [\alpha + m, \beta + n; ze^{2\pi i(m\beta - n\alpha)}]$$

if $m, n \in \mathbf{Z}$.

The main computational tool for Chern-Simons invariants in [KK3] is the following theorem.

2.10 Theorem. *Let X be a 3-manifold with $\partial X = T^2$. Let $\rho_t : I \rightarrow \chi(X)$ be a path of representations. Let $(\alpha(t), \beta(t))$, $t \in I$ be a path in $\mathbf{R}^2$ so that $\rho_t(\mu) = e^{2\pi i \alpha(t)}$ and $\rho_t(\lambda) = e^{2\pi i \beta(t)}$. If*

$$c_X(\rho_0) = [\alpha(0), \beta(0), z_0]$$

and

$$c_X(\rho_1) = [\alpha(1), \beta(1), z_1],$$

then

$$z_1 z_0^{-1} = \exp(2\pi i \int_0^1 \alpha d\beta - \beta d\alpha).$$

Again this theorem says that for a manifold X with torus boundary, the “difference” in Chern-Simons invariants of 2 representations can be calculated solely by knowing the map $\chi(X) \rightarrow \chi(T)$.

We will give an example below which shows how easy it is to compute Chern-Simons invariants with this theorem. It contains Theorem 2.5 as a special case. Before we compute, however, we would like to point out a more theoretical interpretation of this result and draw some (perhaps surprising) conclusions about the subvariety $\text{Image}(\chi(X) \rightarrow \chi(T))$.

The following is essentially a restatement of the previous theorem.

2.11 Corollary.

1. *The connection 1-form*

$$-2\pi i(\alpha d\beta - \beta d\alpha)$$

on the trivial principal S^1 bundle $\mathbf{R}^2 \times S^1 \rightarrow \mathbf{R}^2$ descends to give an orbifold connection 1-form ω on $L_T \rightarrow \chi(T)$. Given any 3-manifold with boundary T , the lift $c_X : \chi(X) \rightarrow L_T$ of the restriction $\chi(X) \rightarrow \chi(T)$ is parallel with respect to this connection.

2. *The symplectic form on $\mathbf{R}^2$,*

$$-4\pi i(d\alpha \wedge d\beta)$$

pushes down to give the curvature $F(\omega)$ of ω .

The proof of this corollary is just an application of the fact that if ω is a connection in an S^1 bundle, and γ is a loop, then

$$\text{hol}_\omega(\gamma) = \exp(2\pi i \int_0^1 \gamma^*(\omega)).$$

The statement of this corollary gives strong restrictions to what the image of $\chi(X) \rightarrow \chi(T)$ can be, and we illustrate this fact now. The following argument was shown to me by Chris Herald.

Suppose that $\chi(X)$ contains a loop. (For example, if X is the exterior of the trefoil knot, then there is a loop consisting of the arc of non-abelian representations together with part of the arc of abelian representations, see figure 4. Likewise, if X is the exterior of the figure 8 knot, then $\chi^*(X)$ contains a smooth circle; see figure 5.) Let $\gamma : I \rightarrow \chi(X)$ be a loop and consider its image in $\chi(T)$. By Stoke’s theorem the symplectic area which γ bounds in $\chi(T)$ is equal to $\int_0^1 \gamma^*(\omega)$, since $d\omega$ is the push forward of the area form in $\mathbf{R}^2$. But since the loop lies in $\chi(X)$, the Chern-Simons function c_X gives a parallel lift of γ to L_T . Hence the holonomy $\text{hol}_\omega(\gamma)$ is trivial, i.e. is equal to

Id in $SU(2)$. Recall that the formula for the holonomy is just $e^{2\pi i \int \gamma^*(\omega)}$. Hence we conclude that the image of γ in $\chi(T)$ must bound zero area mod $\mathbf{Z}$.

This puts constraints on the map $\chi(X) \rightarrow \chi(T)$. Figure 8 shows some forbidden examples. (conjugacy classes of) representations:

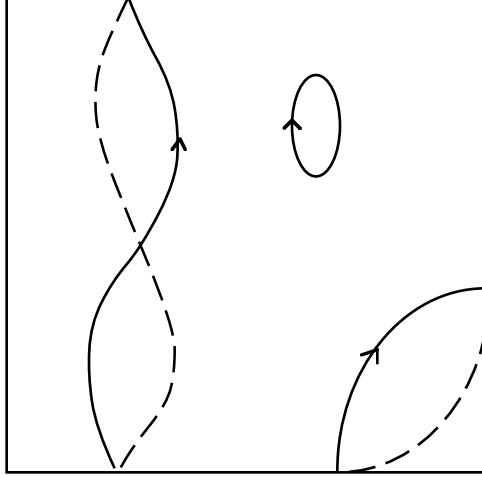


Figure 8.

Symplectic subvarieties like $\chi(X) \rightarrow \chi(\Sigma)$ which have this extra structure, namely a parallel lift to L_Σ , are called *Legendrian*. In [He] Herald carries out a detailed analysis of perturbations of the maps $\chi(X) \rightarrow \chi(\Sigma)$ and c_X similar to Taubes' [T] and Floer's [F] perturbations of the Chern-Simons function for closed manifolds. Among other things he shows that the image of $\chi(X) \rightarrow \chi(\Sigma)$ varies by a Legendrian cobordism.

Both Theorems 2.8 and 2.10 generalize to other Lie groups. An especially interesting case is to take $G = SL(2, \mathbf{C})$. For the special case when X is a cusped hyperbolic manifold, these theorems (for $SL(2, \mathbf{C})$) can be reinterpreted as a result of Yoshida's [Y1] constructing an analytic function on $\chi_{SL(2, \mathbf{C})}(X)$ near the complete hyperbolic representation (see [KK3]) whose real and complex parts correspond to the hyperbolic volume and the Chern-Simons invariant of the Levi-Civita connection. The theorems also work just as well for links as for knots.

2.12 A Computation

We finish this section with a piece of the computation of the Chern-Simons invariant of a graph manifold obtained by gluing $X = \Sigma(a_1, \dots, a_m) - N(\text{regular fiber})$ to $Y = \Sigma(c_1, \dots, c_n) - N(\text{regular fiber})$ along their boundary torus. Here $\Sigma(a_1, \dots, a_m)$ denotes the Seifert Fibered homology sphere with singular fibers of multiplicity a_i . We will not carry out the entire computation, but only one part to illustrate using the inner product $\langle \cdot, \cdot \rangle : L_T \times L_T \rightarrow S^1$.

Consider first X . Let $a = a_1 \cdots a_m$ be the products of the multiplicities. Now

$$\pi_1 X = \langle x_1, \dots, x_M, h \mid h \text{ central}, x^{a_i} h^{b_i} = 1 \rangle$$

for some integers b_i satisfying

$$a \sum_{i=1}^m \frac{b_i}{a_i} = 1.$$

Moreover the meridian and longitude are given by $\mu = x_1 \cdots x_m$, $\lambda = h$

Suppose that $\rho_X : \pi_1 X \rightarrow SU(2)$ is a non-abelian representation. Then from the presentation of $\pi_1 X$ we see that each x_i must go to a $2a_i^{th}$ root of unity in $SU(2)$ (since h must be sent to $\pm Id$). Let l_i be the integer between 0 and a_i so that x_i is sent to a conjugate of $\exp \frac{2\pi i l_i}{2a_i}$, and let

$$e_X = a \sum_{i=1}^n \frac{l_i}{a_i}.$$

Then, using Theorem 2.10 and using the simple description of $\chi(X)$ in terms of linkages as in [KK1] one computes $c_X(\rho_X)$ to be

$$[\alpha, \beta; e^{-2\pi i(e_X^2/4a + \beta\alpha)}]$$

where $\rho_X(\mu) = e^{2\pi i\alpha}$ and $\rho_X(\lambda) = e^{2\pi i\beta}$. (notice that $\beta \in \mathbf{Z}[\frac{1}{2}]$ since h is central.)

Likewise,

$$\pi_1 Y = \langle y_1, \dots, y_M, k \mid k \text{ central}, y^{c_i} h^{d_i} = 1 \rangle.$$

If $\rho_Y \rightarrow SU(2)$ is a representation we have rotation numbers for Y and define e_Y similarly. This time we use the meridian and longitude $\bar{\mu} = y_1 \cdots y_n$ and $\bar{\lambda} = k$. Letting $\rho_Y(\bar{\mu}) = e^{2\pi i\bar{\alpha}}$ and $\rho_Y(\bar{\lambda}) = e^{2\pi i\bar{\beta}}$ we get

$$c_Y(\rho_Y) = [\bar{\alpha}, \bar{\beta}; e^{-2\pi i(e_Y^2/4a + \bar{\beta}\bar{\alpha})}].$$

Now suppose that we are given a gluing map $\phi : \partial X \rightarrow \partial Y$, expressed in terms of the bases μ, λ and $\bar{\mu}, \bar{\lambda}$ by $\phi(\mu) = u\bar{\mu} + w\bar{\lambda}$ and $\phi(\lambda) = v\bar{\mu} + z\bar{\lambda}$ (so $uz - vw = -1$).

Let $T = \partial X$. Suppose that $\rho_{X|T}$ equals $\rho_{Y|T}$. Then these two representations glue together to give a representation ρ_M of M , where $M = X \cup_\phi Y$. Theorem 2.8 now implies that

$$c_M(\rho_M) = \langle c_X(\rho_X), c_Y(\rho_Y) \rangle = \langle [\alpha, \beta; e^{-2\pi i(e_X^2/4a + \beta\alpha)}], [\bar{\alpha}, \bar{\beta}; e^{-2\pi i(e_Y^2/4a + \bar{\beta}\bar{\alpha})}] \rangle.$$

We must be careful since c_X and c_Y are expressed in terms of different bases, namely bases differing by the linear map ϕ . We will express everything in terms of the bases $\bar{\mu}$ and $\bar{\lambda}$.

Write $\alpha = u\bar{\alpha} + w\bar{\beta}$ and $\beta = v\bar{\alpha} + z\bar{\beta}$. Substituting, we obtain:

$$\begin{aligned} c_X(\rho_X) &= [\alpha, \beta; e^{-2\pi i(\frac{e_X^2}{4a} + \alpha\beta)}] \\ &= [u\bar{\alpha} + w\bar{\beta}, v\bar{\alpha} + z\bar{\beta}; e^{-2\pi i(\frac{e_X^2}{4a} + (u\bar{\alpha} + w\bar{\beta})(v\bar{\alpha} + z\bar{\beta}))}]. \end{aligned}$$

Since β and $\bar{\beta}$ are half integers it follows that α and $\bar{\alpha}$ are rational with denominator $2v$.

Write

$$\bar{\alpha} = \frac{p}{2v}, \quad \bar{\beta} = \frac{\kappa}{2}$$

for some integers p and κ . Then

$$c_Y(\rho) = [\frac{p}{2v}, \frac{\kappa}{2}; e^{-2\pi i(\frac{e_Y^2}{4c} + \frac{p}{2v}\frac{\kappa}{2})}]$$

and

$$c_Y(\rho) = [\frac{p}{2v}, \frac{\kappa}{2}; e^{-2\pi i(\frac{e_X^2}{4a} + (u\bar{\alpha} + w\bar{\beta})(v\bar{\alpha} + z\bar{\beta}))}].$$

We can now take the inner product:

$$\begin{aligned} c_M &= -(\frac{e_X^2}{4a} + (u\bar{\alpha} + w\bar{\beta})(v\bar{\alpha} + z\bar{\beta})) - (\frac{e_Y^2}{4c} + \bar{\alpha}\bar{\beta}) \\ &= -\frac{e_X^2}{4a} - \frac{e_Y^2}{4c} - \bar{\alpha}\bar{\beta}((1 + uz + vw)) - \bar{\alpha}^2 uv - \beta^2 wz \\ &= -\frac{e_X^2}{4a} - \frac{e_Y^2}{4c} - \frac{p^2 u}{4v} - \frac{w\kappa^2}{4}(2p + z). \end{aligned}$$

You will notice the similarities of the first two terms with Fintushel and Stern's formula for Seifert Fibered homology spheres; the last two terms are “interaction” terms defined solely in terms of the gluing map and the restriction of the representation to the separating torus.

3 Spectral Flow

3.1 The Hessian of the Chern-Simons Function

In this last lecture we investigate the spectral flow of the Hessian of the Chern-Simons function (and the related Atiyah-Patodi-Singer odd signature operator) along a path of connections on a 3-manifold. We will describe a method which allows us to compute the spectral flow for Dehn surgery on knots in terms of $\chi(X_K)$ and its image in $\chi(T)$.

We begin by defining the Hessian of the Chern-Simons function, following Taubes [T]. Consider the space $\mathcal{B} = \mathcal{A}/\mathcal{G}$ of gauge equivalence classes of connections on a *closed* 3-manifold M . By

completing $\mathcal{A}, \mathcal{G}$ in appropriate Sobolev norms we can give $\mathcal{B}$ the structure of a (singular) infinite-dimensional Banach manifold. (We will ignore all technicalities about completions here; the reader can look up any one of many good references, e.g. [DK]. Our only intention is to display the analogy with Morse theory). Since $\mathcal{A}$ is an affine space modeled on $\Omega_M^1 \otimes su(2)$, the tangent space to $\mathcal{A}$ at A is $T_A \mathcal{A} = \Omega_M^1 \otimes su(2)$. We wish to describe the tangent space to $\mathcal{B}$. Since $\mathcal{B} = \mathcal{A} / \mathcal{G}$, the pull back of the tangent space of $\mathcal{B}$ to $\mathcal{A}$ can be identified with the orthogonal complement to the tangent space of the $\mathcal{G}$ -orbits.

Now $\mathcal{G} = Maps(M, SU(2))$ and so $T_1 \mathcal{G} = Maps(M, su(2)) = \Omega_M^0 \otimes su(2)$. The function $\mathcal{G} \rightarrow \mathcal{A}$ taking g to $g \cdot A$ maps $\mathcal{G}$ onto the orbit of A and its differential at $1 \in \mathcal{G}$ is just the covariant derivative $d_A : \Omega_M^0 \otimes su(2) \rightarrow \Omega_M^1 \otimes su(2)$. Now $\mathcal{A}$ has a Riemannian metric induced by the L^2 inner product

$$\langle a, b \rangle = - \int_M tr(a \wedge *b), \quad a, b \in T_A \mathcal{A} = \Omega_M^1 \otimes su(2).$$

Therefore the pullback of $T_* \mathcal{B}$ to $\mathcal{A}$ is the $\mathcal{G}$ -equivariant subbundle whose fiber over $A \in \mathcal{A}$ is

$$(Im d_A : \Omega_M^0 \otimes su(2) \rightarrow \Omega_M^1 \otimes su(2))^\perp = ker d_A^* : \Omega_M^1 \otimes su(2) \rightarrow \Omega_M^0 \otimes su(2).$$

(This is not quite a bundle, since it “jumps up” along the reducibles, i.e. along the set of connections A in $\mathcal{A}$ for which $ker d_A : \Omega_M^0 \otimes su(2) \rightarrow \Omega_M^1 \otimes su(2)$ is non-zero. However we can think of $\mathcal{B}$ as being stratified by the dimension of the centralizer of the holonomy of a connection, and then along each open stratum this gives the pullback of the tangent space to $\mathcal{B}$.)

The Chern-Simons function $cs : \mathcal{B} \rightarrow S^1$ lifts to $cs : \mathcal{A} \rightarrow \mathbf{R}$ (in a given trivialization) using the equation (2.1). Now if $A \in \mathcal{A}$, $B \in T_A \mathcal{A} = \Omega_M^1 \otimes su(2)$, then

$$\begin{aligned} dcs_A(B) &= \lim_{t \rightarrow 0} \frac{1}{t} (cs(A + tB) - cs(A)) \\ &= \frac{1}{8\pi^2} \int_M tr(dA \wedge B + dB \wedge A + \frac{2}{3}(A \wedge A \wedge B + A \wedge B \wedge A + B \wedge A \wedge A)) \\ &= \frac{1}{8\pi^2} \int_M tr(dA \wedge B + dB \wedge A + 2A \wedge A \wedge B) \\ &= \frac{1}{4\pi^2} \int_M tr((dA + A \wedge A) \wedge B) \\ &= \frac{1}{4\pi^2} \langle *F(A), B \rangle. \end{aligned}$$

(Stokes' theorem is used to show that $\int tr(dB \wedge A) = \int tr(B \wedge dA)$.) Thus the gradient of the Chern-Simons function is $grad cs(A) = \frac{1}{4\pi^2} *F(A)$.

This is also the pullback of the gradient of $cs : \mathcal{B} \rightarrow S^1$ to $\mathcal{A}$ since the Bianchi identity implies that $d_A^*(F(A)) = - * d_A F(A) = 0$ and so $*F(A)$ lies in the subbundle whose fiber over A is $ker d_A^*$. Thus the set of critical points of $cs : \mathcal{B} \rightarrow S^1$ is just $\mathcal{F} / \mathcal{G} = \chi(M)$.

Continuing the Morse theory analogy we need a notion of index of a critical point. In finite dimensions this is defined to be the signature of the Hessian at a critical point. Let us first calculate the Hessian of cs .

The Hessian is the linearization of the gradient. Since $grad\ cs : \mathcal{A} \rightarrow T_*\mathcal{A}$ is given by $A \mapsto \frac{1}{4\pi^2} * F(A)$, and $F(A + tB) = F(A) + td_AB + t^2 B \wedge B$ for $B \in \Omega_M^1 \otimes su(2)$, we compute that the Hessian of cs at A is given by

$$H_A : T_A \mathcal{A} \rightarrow T_A \mathcal{A}, \quad B \mapsto *d_A B.$$

This is the Hessian if we view cs as a function from $\mathcal{A}$ to $\mathbf{R}$. To compute the Hessian if we consider cs as an S^1 -valued function on $\mathcal{B}$, we can use the connection ∇ on $\pi^*(T_*\mathcal{B})$ obtained from the trivial connection on $T_*\mathcal{A} = \mathcal{A} \times \Omega_M^1 \otimes su(2)$. Thus

$$H_A = \nabla grad\ cs_A = proj_{\ker d_A^*} * d_A : \ker d_A^* \rightarrow \ker d_A^*$$

where $\pi^*(T_A \mathcal{B}) = \ker d_A^*$, and $proj_{\ker d_A^*}$ is the orthogonal projection in the fiber $T_A \mathcal{A}$, (which is just the L^2 projection $\Omega_M^1 \otimes su(2) \rightarrow \ker d_A^*$). At a critical point, $F(A) = d_A d_A = 0$, and so $d_A^*(d_A) = 0$. Thus the Hessian at a critical point $A \in \chi(X)$ is just $*d_A$. It is easy to compute that H_A is self-adjoint with respect to the L^2 inner product.

3.2 Spectral Flow

What distinguishes H_A from the Hessian of a function on a finite-dimensional manifold is that H_A has infinitely many positive and negative eigenvalues. However, what is relevant in Morse theory is not so much the signature of the Hessian at a critical point, but the difference in signatures of the Hessian at two different critical points.

Consider the finite dimensional case. If Z is a finite dimensional manifold and $f : Z \rightarrow \mathbf{R}$ is a Morse function, let $grad\ f$ denote its gradient vector field. If ∇ is a connection in the tangent bundle then $\nabla(grad\ f)_z : T_z Z \rightarrow T_z Z$ gives a family of self-adjoint endomorphisms parameterized by the points of Z . In particular, if z_0, z_1 are non-degenerate critical points, (so $\nabla(grad\ f)_{z_i}$ is invertible, $i = 0, 1$), and z_t is a path in Z from z_0 to z_1 , then the eigenvalues of $\nabla(grad\ f)_{z_t}$ vary continuously with t . If $\lambda_i(t), i = 1, \dots, \dim Z$ denotes the eigenvalues, then the quantity

$$\#\{i \mid \lambda_i(0) < 0, \lambda_i(1) > 0\} - \#\{i \mid \lambda_i(0) > 0, \lambda_i(1) < 0\}$$

is equal to the difference in the Morse index of f at z_1 and z_0 and is called the *Spectral Flow* of the family of self-adjoint operators $\nabla(grad\ f)_{z_t}$.

This same quantity makes sense for a continuous family of self-adjoint operators with discrete spectrum (with each eigenvalue of finite multiplicity). We will show below that H_A has such a spectrum, and so if $\rho_0, \rho_1 \in \chi(M)$, we define the *spectral flow from ρ_0 to ρ_1* to be the spectral flow of the family H_{A_t} where A_t is a path of connections with A_t flat with holonomy ρ_i for $i = 0, 1$. It is an integer.

There are several technicalities which we have ignored. Firstly, we have assumed that the path A_t lies in a single stratum of $\mathcal{B}$, so that $T_*\mathcal{B}$ does not jump up along the path. Second, we have not shown that H_A has a discrete and continuously varying spectrum. We are also blurring the distinction between representations and characters (and between connections and gauge-equivalence classes of connections) and, in particular, we should know what changing A to $g \cdot A$ does to the spectral flow. Finally, we would like to know how the choice of path A_t affects the spectral flow.

Taubes [T] introduces the following trick to deal with all these problems. If $A \in \mathcal{A}$ is a connection, decompose $\Omega^0 \otimes su(2) \oplus \Omega^1 \otimes su(2)$ into

$$\Omega^0 \otimes su(2) \oplus \text{Image } d_A \oplus \ker d_A^*.$$

Let $B_A : \Omega^0 \otimes su(2) \oplus \text{Image } d_A \oplus \ker d_A^* \longrightarrow \Omega^0 \otimes su(2) \oplus \text{Image } d_A \oplus \ker d_A^*$ be the self-adjoint operator given by the matrix:

$$B_A = \begin{pmatrix} 0 & d_A & 0 \\ d_A^* & 0 & 0 \\ 0 & 0 & H_A \end{pmatrix}.$$

Then

1. The spectrum of the top left block $\begin{pmatrix} 0 & d_A \\ d_A^* & 0 \end{pmatrix}$ is symmetric since if $\begin{pmatrix} \phi \\ \tau \end{pmatrix}$ is a λ -eigenvector then $\begin{pmatrix} \phi \\ -\tau \end{pmatrix}$ is a $-\lambda$ -eigenvector.
2. If A is flat, then $H_A \tau = *d_A \tau$, and so

$$\begin{aligned} B_A(\phi, \tau) &= (d_A^*(\text{proj}_{\text{Im } d_A} \tau), d_A \phi + H_A(\text{proj}_{\ker d_A^*} \tau)) \\ &= (d_A^* \tau, d_A \phi + *d_A \tau). \end{aligned}$$

For a general connection A , let $D_A(\phi, \tau) = (d_A^* \tau, d_A \phi + *d_A \tau)$ acting on $\Omega^0 \otimes su(2) \oplus \Omega^1 \otimes su(2)$. Then $D_A = B_A$ if A is flat. Moreover, D_A is elliptic, and $D_A - B_A$ is compact.

Therefore, the spectrum of B_A is discrete, and so also of H_A . The spectral flow of H_{A_t} equals the spectral flow of B_{A_t} along any path in $\mathcal{A}^*$. If $A_0, A_1 \in \mathcal{A}^*$ are flat then $SF(B_{A_t}) = SF(D_{A_t})$. However, the domain of the operator D_{A_t} is independent of t ; it is just the image of L_1^2 in L^2 . In particular, the spectral flow of D_{A_t} makes sense for any path A_t between any two connections A_0

and A_1 . Thus we lose nothing by taking the spectral flow of the family D_{A_t} instead of the family H_{A_t} , in fact we gain since D_{A_t} makes sense even when A_t does not lie in one stratum of $\mathcal{B}$.

This leaves only the question of the dependence on the path and the choice of gauge equivalence class of connections. This is done by showing that D_A is the tangential operator (in the sense of [APS]) of the self-duality operator on a 4-manifold. What we must compute is the spectral flow of a family around a loop in $\mathcal{B}$, or equivalently, along a path A_t in $\mathcal{A}$ from A to $g \cdot A$. Applying the index theorem of [APS] we conclude that the spectral flow along any such loop is equal to the index of the self-duality operator on a bundle over $M \times S^1$, and this index is divisible by 8. Thus the spectral flow of the family D_{A_t} is well-defined mod 8 as a function on the pairs $\rho_0, \rho_1 \in \chi(X)$. (see the appendix to [KKR] for details.)

Note that the operator D_t is a disguised form of the Atiyah-Patodi-Singer odd signature operator, i.e. half of the tangential operator of the signature operator [APS].

Finally let us mention the definition of Floer's instanton homology. First, the chain complex is generated by the points of $\chi(M)$, and is $\mathbf{Z}/8$ graded by taking the grading of ρ to be the spectral flow from ρ to some fixed ρ_0 . The differentials in the chain complex are more difficult to explain, and as of yet I do not know of a rigorous way to obtain the differentials from the "pillowcase" picture, although several people have told me this should be possible, ostensibly by identifying the instanton Chain complex with Floer's "symplectic" homology. I will have nothing to say about these differentials in what follows (except to point out when they must be zero!)

Among the earliest computations of spectral flow were Fintushel and Stern's computations for Seifert fibered homology spheres [FS]. Their computations were carried out by showing that given any Seifert fibered homology spheres M and $\rho \in \chi(M)$, there exists a 4-manifold Z such that the boundary of Z equals $-M \cup L$ where L is a union of lens spaces, and ρ extends to a representation of $\pi_1 Z$. Applying the Atiyah-Patodi-Singer index theorem (and the relationship between the spectral flow and the η invariant of [APS]) the computation reduces to a computation of the Atiyah-Patodi-Singer ρ_α invariants of lens spaces (which is easy since they have finite fundamental groups) and the Chern-Simons invariants of M .

Finding such a flat cobordism is rare. Other computations using this approach include [KKR] for certain graph manifolds and abelian representations, and [SS] for certain links of algebraic singularities.

Let us interpret several of the computations of [FS] for Dehn surgery on knots in terms of the image $\chi(X_K) \rightarrow \chi(T)$.

3.3 Two Surgeries on the Trefoil

A. We start with the simplest non-trivial example: $+1$ surgery on the Trefoil; $S^3(K, +1)$, which is just the Poincaré homology sphere $\Sigma(2, 3, 5)$. Figure 9 Shows $\chi(X_K)$ and $\chi(N(K))$, on the left in the natural S^3 meridian and longitudes μ, λ and on the right in the meridian and longitudes $\bar{\mu}, \bar{\lambda}$ in $\Sigma(2, 3, 5)$. Thus $\bar{\mu} = \mu + \lambda$ and $\bar{\lambda} = \lambda$. (Recall from section 1.5 that $\chi^*(X_K)$ is parameterized by $\Phi(t, -6t + \frac{1}{2})$, $t \in (\frac{1}{12}, \frac{5}{12})$.)

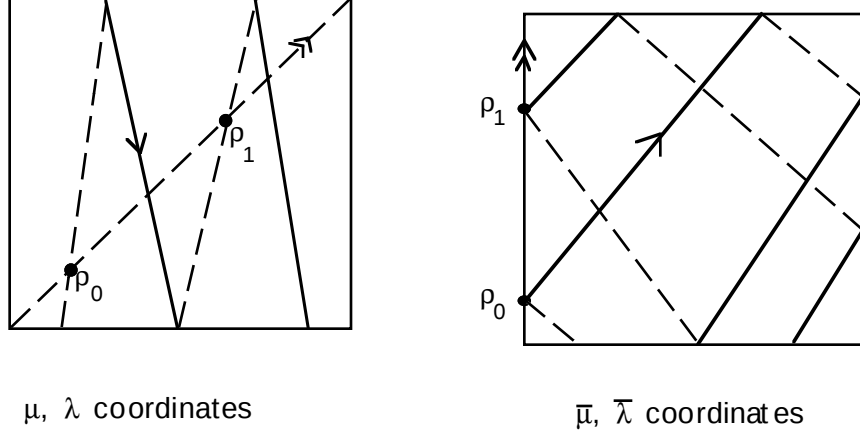


Figure 9.

Fintushel and Stern compute that in this example, $SF(\rho_0, \rho_1; \Sigma(2, 3, 5)) \equiv 4 \pmod{8}$.

B. Consider a slightly more complicated case, $\frac{1}{3}$ surgery on the trefoil. Figure 6 has the pillowcase picture in the S^3 longitude and meridian. Taking $\bar{\mu} = \mu + 3\lambda$ and $\bar{\lambda} = \lambda$ we obtain the following figure in the natural coordinates $\bar{\mu}$ and $\bar{\lambda}$ for $S^3(K, \frac{1}{3}) = \Sigma(2, 3, 17)$:

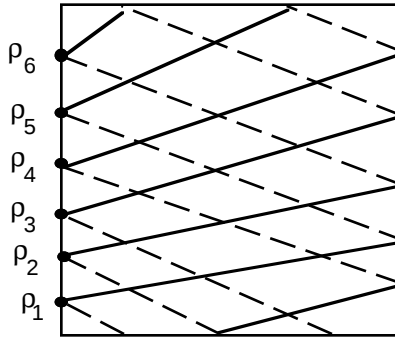


Figure 10.

Fintushel and Stern's computations imply:

$$SF(\rho_1, \rho_5) = 4, SF(\rho_4, \rho_5) = 6, SF(\rho_4, \rho_2) = 2, SF(\rho_2, \rho_6) = 4, \text{ and } SF(\rho_6, \rho_3) = 6.$$

These examples will be used in the next section to derive a formula for the spectral flow.

3.4 The Characteristic Cohomology Class

Consider the 3-manifold pair $(N, T) = (D^2 \times S^1, S^1 \times S^1)$ and give T the coordinates $\mu = S^1 \times *$ and $\lambda = * \times S^1$. Let $\hat{\chi}(T) = \chi(T) - \{\text{central representations}\}$; this is the pillowcase with the corners removed. Let $\hat{\chi}(N) = \chi(N) - \{\text{central representations}\}$; this is an open arc mapping to the left vertical edge of the pillowcase. Then define the *Characteristic Cohomology Class* for (N, T) ,

$$s_{(N,T)} \in H^1(\hat{\chi}(T), \hat{\chi}(N); \mathbf{Z}/8)$$

to be the Poincaré dual to the homology class z indicated in Figure 11.

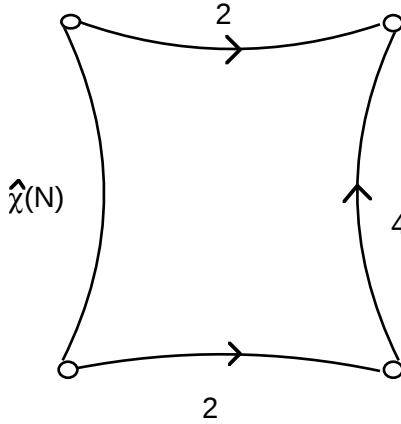


Figure 11.

In this figure the cohomology class z equals 2 times the top horizontal edge + 2 times the bottom horizontal edge + 4 times the right vertical edge (oriented as indicated). Thus, given a path γ in $\hat{\chi}(T)$ with endpoints in $\hat{\chi}(N)$, $s_{(N,T)}(\gamma) = \gamma \cdot z \pmod{8}$.

Now the important observation which the reader should verify is that in the previous two examples, letting γ be a path in $\chi(X_K)$ joining any pair $\rho_0, \rho_1 \in \chi(M)$ (where M denotes $\Sigma(2, 3, 5)$ in the first example and $\Sigma(2, 3, 17)$ in the second example),

$$SF(\rho_0, \rho_1) = s_{(N,T)}(\gamma).$$

This suggests that the spectral flow between two representations of a surgery on a knot K can be computed by understanding the image of $\chi(X_K)$ in the pillowcase and applying the class $s_{(N,T)}$.

This is almost true, but one more invariant must be introduced, namely the *Maslov Index*. For simplicity we introduce it for the pillowcase only.

Let $\mathcal{L}$ be the vertical line field on $\hat{\chi}(T)$, i.e. the image of the vertical line field in $\mathbf{R}^2 - (\frac{1}{2}\mathbf{Z})^2$ under the projection $\Phi : \mathbf{R}^2 \rightarrow \chi(T)$. Given an oriented, *immersed* curve τ in $\hat{\chi}(T)$ whose endpoints

are transverse to $\chi(\hat{N})$, the Maslov index is defined to be the winding number of the tangent vector $\dot{\tau}$ with respect to $\mathcal{L}$, and is denoted by $\gamma(\tau)$. Figure 12 Shows how to compute $\gamma(\tau)$; it picks up a $+1$ or -1 each time τ is tangent to $\mathcal{L}$; $+1$ if the tangency is on the right and -1 if the tangency is on the left of τ .

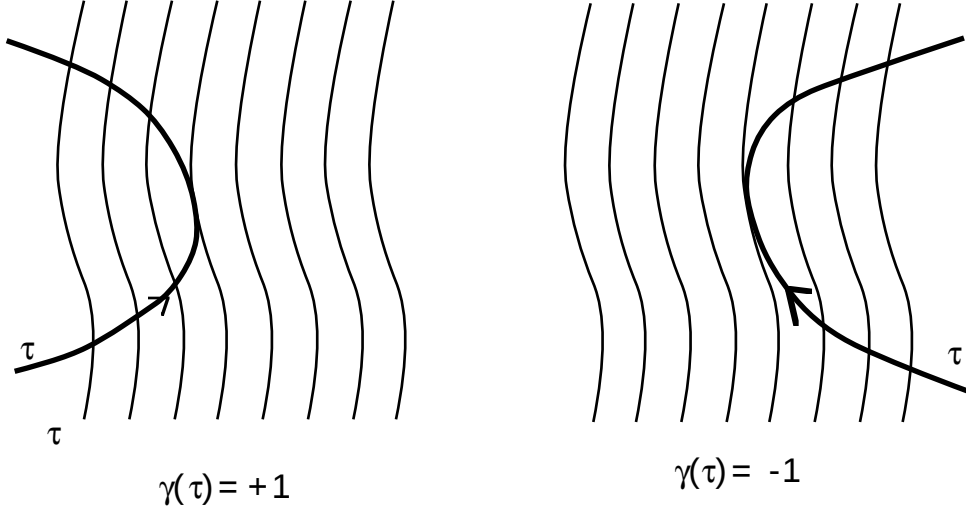


Figure 12.

Then we have:

3.5 Theorem. *Let $K \subset M$ be a knot, and let $\rho_0, \rho_1 \in \chi^*(M)$ be two representations. Suppose there exists a path $\rho : I \rightarrow \chi^*(X_K)$ from ρ_0 to ρ_1 , and suppose that $\chi^*(X_K)$ is a smooth 1-dimensional variety along ρ , i.e. $H^1(X_K; \text{ad}\rho_t)$ is 1-dimensional for all $t \in I$. Finally suppose that the image of ρ_t in the pillowcase lies in $\hat{\chi}(T)$ and is transverse to $\mathcal{L}$ at the endpoints (i.e. $H^1(M; \text{ad}\rho_i) = 0$ for $i = 0, 1$.) Let τ_t denote the image of ρ_t in the pillowcase. Then:*

$$SF(\rho_0, \rho_1) = s_{(N,T)}(\tau_t) + \gamma(\tau_t).$$

Remarks: 1. This theorem was first proven by Yoshida [Y3] using his splitting theorem [Y2] for spectral flow. Other proofs of the splitting theorem are now available; the clearest one in my opinion is in [MW]. See also [CLM] and [N].

2. The assumptions on $\chi^*(X_K)$ and $\chi(M)$ are “generic”, i.e. they are moduli spaces of the dimension predicted by the index theorem. In [T],[F], a family of perturbations called “geometric perturbations” of the Chern-Simons function $\mathcal{B} \rightarrow S^1$ are given which ensure that the critical points are generic. Chris Herald [He] works this out carefully for a manifold with boundary, and in particular proves that a geometric perturbation can be found which makes $\chi^*(X_K)$ a smooth

1-dimensional variety. In principle his methods can be used to give the correct input to this theorem. One needs to compute “perturbed” representation spaces, i.e. $\chi_p(M)$ and $\chi_p(X_K)$ where p denotes some perturbation of the Chern-Simons function and χ_p the corresponding set of critical points. This involves additional difficulties; examples need to be worked out. However, finding good perturbations is a topology and algebra problem; it involves no analysis. One needs to find an appropriate link in the manifold and investigate the character variety of the link exterior.

3. There is nothing special about the line field $\mathcal{L}$; any line field can be used. However, choosing a different line field will change $s_{(N,T)}$. It is the sum of the two which must remain invariant. We have chosen $\mathcal{L}$ for convenience. Thus the notation $s_{(N,T)}$ is not precise; strictly speaking this class depends on $\mathcal{L}$. It is the sum $s_{(N,T)}(-) + \gamma(\mathcal{L}, -)$ which is independent of $\mathcal{L}$.

4. We assume that the image $\chi^*(X_K) \rightarrow \chi(T)$ misses the corners of the pillowcase. This is OK if M is a surgery on a knot in S^3 , as we have observed in section 1.4. In general, however, we need to understand what happens if the image $\chi(X_K) \rightarrow \chi(T)$ passes through a corner of the pillowcase. The class $s_{(N,T)}$ no longer makes sense and needs to be replaced by a more general object, maybe some kind of intersection cohomology class which keeps track of the strata. What is at issue is that the dimension of $H^1(T; \text{ad}\rho)$ jumps up from 2 to 6 dimensions at the corners. Thus the Zariski tangent space of $\chi(T)$ is not a vector bundle near the corners, but a sheaf. One could conjecture that the spectral flow through a corner depends on the angle at which the path approaches the corner.

5. In the each of the two examples above, the Maslov index term is zero, i.e. the image $\chi(X_K) \rightarrow \chi(T)$ is transverse to the vertical line field $\mathcal{L}$. It is unlikely that this is true in general; there are probably examples of surgeries on 2-bridge knots where this fails. A more subtle question is whether $\chi(X_K)$ is transverse to $\mathcal{L}$ at its endpoints when X_K is the complement of a knot in M . This is closely related to the question of whether there exist a representation ρ of closed 3-manifolds M which is an isolated point in $\chi(M)$ but for which $H^1(M; \text{ad}\rho) \neq 0$. One conjecture states that this is not possible.

6. It should be possible to generalize the characteristic class $s_{(N,T)}$ to any 3-manifold pair (X, Σ) , and for that matter any compact Lie group G and representation instead of just $SU(2)$ and the adjoint representation. There are many difficulties in doing this, related to the singularities of $\chi_G(X)$ and $\chi_G(\Sigma)$. See the method in [KK4] for constructing $s_{(N,T)}$ and Theorem 1.7 in [KK5] for the technical result needed at least for the case when one stays within a stratum of $\chi(\Sigma)$. This makes the pair $(H^1(\chi(X), \chi(\Sigma); \mathbf{Z}/n); s_{(X,\Sigma)})$ look suspiciously like a TQFT.

We note here that the *computation* of the coefficients of $s_{(N,T)}$ for $G = SU(2)$ and the adjoint representation depends on Fintushel-Stern’s computations. It turns out that the computation for

$SU(2)$ and the defining representation (i.e. $\mathbf{C}^2$) looks like figure 11 except that in that case $s_{(N,T)}$ is a mod 2 class, the 2 is changed to a 1 and the 4 to a 0. One can find the appropriate coefficients for any representation of $SU(2)$

This theorem works well for surgeries on torus knots and 2-bridge knots, or more generally, any Brieskorn sphere $\Sigma(p, q, r)$. The inspired reader should work out the example of $\frac{1}{4}$ surgery on the figure 8 knot using Figure 5 (and changing coordinates to $\bar{\mu} = \mu + 2\lambda$, $\bar{\lambda} = \lambda$) to test whether s/he has understood the statement of Theorem 3.4. It turns out again that $\gamma = 0$, and so $SF(\rho_i, \rho_j)$ is always even. Hence the boundary operators in the Instanton chain complex are all zero and so the Instanton chain complex equals its homology. Up to a $\mathbf{Z}/8$ cyclic permutation the Instanton homology is $(\mathbf{Z}^2, 0, \mathbf{Z}^2, 0, \mathbf{Z}^2, 0, \mathbf{Z}^2, 0)$.

3.6 The Non-Generic Case So far we have been discussing spectral flow for Dehn surgeries and have seen that in certain *generic* situations, the entire picture is contained in the image $\chi(X_K) \rightarrow \chi(T)$. We alluded to perturbations which make $\chi(X_K)$ generic. We now want to describe an alternative method, which does not suppose that $\chi(X_K)$ is generic. Along the way we will define spectral flow for a manifold with boundary. (Notice that H_A and D_A are not self-adjoint for manifolds with non-empty boundary unless some boundary conditions are imposed.) This will lead to interesting formulas for spectral flow in terms of certain Massey products in the twisted cohomology of X . Moreover, a new feature is introduced, namely the *analytic structure* of $\chi(X)$ is used to control eigenvalues and eigenvectors. Even for a closed manifold this is interesting; for example Farber and Levine [FL] have used this idea to obtain results about the homotopy invariance of the Atiyah-Patodi-Singer ρ_α invariants on any odd-dimensional manifold.

(Exercise: Show that $L(7, 1)$ and $L(7, 2)$ are not diffeomorphic by computing the spectral flow between corresponding representations using Example 1.5.1 and Theorem 3.5.)

3.7 Analytic families of Flat Connections and Operators

Let X be a 3-manifold with non-empty boundary Σ ; we assume a Riemannian metric on X is given such that the collar of the boundary is isometric to $\Sigma \times I$. A connection A on X is said to be in *cylindrical form* if, on the collar, A is the product of a connection $\hat{A}$ on Σ and the trivial connection in the I direction. The following theorem (which also holds if $\Sigma = \emptyset$) shows that families of flat connections on X can be chosen parameterized by $\text{hom}(\pi_1 X, SU(2))$ which reflect the analytic properties of the algebraic variety $\text{hom}(\pi_1 X, SU(2))$.

3.8 Theorem. *Given a flat connection A with holonomy ρ_A in cylindrical form, there exists a neighborhood $U \subset \text{hom}(\pi_1 X, SU(2))$ of ρ_A and an analytic function $s : U \rightarrow \mathcal{A}$ so that $s(\rho)$ is flat with holonomy ρ for all $\rho \in U$ and $s(\rho)$ is in cylindrical form near the boundary. Here $\mathcal{A}$ is given any Sobolev L_s^2 topology or any C^k topology.*

For a proof see [FKK]. What this theorem says is that we can locally split the holonomy map from flat connections $\mathcal{F}$ to $\text{hom}(\pi_1 X, SU(2))$ in such a way that the splitting s is analytic. Notice that $s : U \rightarrow \mathcal{A}$ is a function from an open set in an analytic (in fact algebraic) variety to an affine space. Since $\mathcal{A}$ is infinite dimensional, some topology must be used before one can define an analytic function, and any Sobolev L_s^2 or C^k topology will do.

3.9 The Easy Case: Closed Manifolds

We show how to use this fact on a closed 3-manifold M where the technicalities of boundary conditions do not enter. Let $\alpha : I \rightarrow \text{hom}(\pi_1 M, SU(2))$ be an *analytic* path of connections. Let

$$D_t : \Omega_M^{0+1} \otimes su(2) \rightarrow \Omega_M^{0+1} \otimes su(2)$$

be the corresponding family of self-adjoint operators

$$D_t(\phi, \tau) = (d_{A_t}^* \tau, d_{A_t} \phi + *d_{A_t} \tau)$$

where $A_t = s(\alpha(t))$, s the splitting of the previous theorem (after perhaps shortening the interval I). Then D_t is an *analytic path of self-adjoint operators* in the sense of [Ka]. The results of analytic perturbation theory imply that the eigenvalues *and eigenvectors* of D_t vary analytically. (Exercise: find a C^∞ path of self-adjoint 2×2 matrices whose eigenvectors do not vary continuously)

The main reason why this method works easily a closed manifold than on a manifold with boundary is that for a closed manifold, the domain of D_A is independent of A ; in fact it is just the image of L_1^2 in L^2 . Making this work on a manifold with boundary is a technical step which we will describe below.

In particular, we have the following principle:

3.10 Principle:. *If $\lambda(t)$ is an eigenvalue of D_t so that $\lambda(0) = 0$, one can tell if $\lambda(t)$ is changing from positive to negative or from negative to positive by looking at the sign and order of the first non-vanishing derivative of $\lambda(t)$ at $t = 0$.*

Returning to our closed manifold M , a basic fact about the operator D_A when A is flat is that the kernel of D_A consists of d_A -harmonic 0- and 1-forms, which by the Hodge and DeRham theorems is isomorphic to the cohomology

$$H^0(M; ad\rho_A) \oplus H^1(M; ad\rho_A).$$

(Here $\rho_A = hol_A \in \chi(M)$.) In particular, the dimension of the kernel of D_A is a cohomological invariant, which means that topologists can compute it if they really want to. Thus along our path of representations α_t we know exactly when the kernel of D_t jumps up, i.e. when an eigenvalue crosses zero.

To get a handle on the spectral flow, we need to know more than just when the kernel jumps, but which way the eigenvalue is crossing through 0. But according to our principle, this is the same as knowing the first non-vanishing derivative of the eigenvalue at the point where it crosses.

This can be encoded abstractly as follows: suppose that $\lambda_i(t), i = 1, \dots, n$ are the paths of eigenvalues which are equal to 0 at $t = 0$. (so $n = \dim H^{0+1}(M; ad\alpha_0)$.) Define a sequence of bilinear forms F_k which are just diagonal forms with the k^{th} derivatives of the λ_i at $t = 0$ along the diagonal. Then the sequence of signatures $\sigma(F_k)$ clearly give the spectral flow through $t = 0$. It turns out these signatures can be computed cohomologically. We describe the first one.

Let $a \in H^1(M; ad\alpha_0) = T_{\alpha_0}\chi(M)$ be the derivative of the path α_t at $t = 0$. It is well known (and easy to prove) that the cup product $[a, a] \in H^2(M; ad\alpha_0)$ equals zero (see e.g. [GM]). Thus we have a “derived ” complex whose chain groups are $H^*(M; ad\alpha_0)$ and whose differential is cupping with a (using the Lie bracket as bilinear form on the coefficients):

$$[a, -] : H^*(M; ad\alpha_0) \longrightarrow H^*(M; ad\alpha_0).$$

The following theorem is proven in [KK4] (and generalized to arbitrary odd dimensions in [KK5])

3.11 Theorem. *In the situation above, let*

$$B_1 : H^1(M; ad\alpha_0) \times H^1(M; ad\alpha_0) \longrightarrow \mathbf{R}$$

be the symmetric bilinear form

$$B_1(x, y) = -[a, x] \cdot y$$

where $[a, \cdot]$ is described above and $\cdot : H^2(M; ad\alpha_0) \times H^2(M; ad\alpha_0) \longrightarrow \mathbf{R}$ is the cup product induced by the bilinear form $su(2) \times su(2) \longrightarrow \mathbf{R}$, $(a, b) \mapsto -tr(ab)$ and the Poincaré duality isomorphism $H^3(M; \mathbf{R}) \cong \mathbf{R}$.

Then the signature of B_1 equals the signature of F_1 . Moreover, if the cohomology of the derived complex $(H^*(M; \text{ad}\alpha_0), [a, -])$ is zero, then the signature of B_1 equals the spectral flow of D_t through $t = 0$.

Thus the first order part of the spectral flow can be computed using cup products, something topologists can do better than computing derivatives of eigenvalues of paths of self-adjoint operators. Notice that B_1 is a homotopy invariant.

The proof of this theorem is quite easy (and standard). One writes down the eigenvalue equation $D_t\phi_t = \lambda_t\phi_t$ and differentiates with respect to t . Then one uses a little algebra to clean up the resulting bilinear form.

Taking higher derivatives leads to Massey products, we will not pursue this here but only say that there is a product (a Massey triple product $(x, y) \mapsto \{a, a, x\} \cdot y$) defined on the cohomology of the derived complex whose signature gives the signature of F_2 . Continuing in this manner one should obtain all the higher signatures of the forms F_k (and hence the spectral flow) as increasingly complicated formulas involving higher order Massey products. In [FL], Farber and Levine circumvent these subtleties by defining a product directly as a torsion pairing over the power series ring from which one can extract (theoretically!) the spectral flow. This immediately leads to the homotopy invariance of the spectral flow, and related results about the homotopy invariance of the Atiyah-Patodi-Singer ρ_α invariants.

3.12 Spectral Flow on a 3-Manifold with Boundary

We turn now to the problem of adapting the results of the last section to a 3-manifold X with non-empty boundary Σ . This is what is needed for examining the spectral flow for Dehn surgeries. As was mentioned, the basic problem is that the domain of D_t must be restricted in order to obtain a self-adjoint operator. As is clear from [Y2], the correct domain to use for gluing problems are analogues of the Atiyah-Patodi-Singer boundary conditions. The technical problem which arises is that *these boundary conditions are changing with t* . Thus some work must be done to apply the results of analytic perturbation theory, namely one must re-parameterize the domains analytically. In [KK4] we were able to do this in an ad-hoc manner when the boundary of X is a torus. A more careful analysis is carried out in [KK5] and so the general problem (in any dimension) can be solved.

We briefly outline what must be done. See [Y2], [KK4],[KK5], and, of course, [APS] for details. First a computation shows that on the collar,

$$D_A = \sigma(\hat{D}_A + \frac{\partial}{\partial u})$$

in the sense of [APS] where the tangential operator $\hat{D}_A$ is the operator on Σ :

$$\hat{D}_A : \Omega_{\Sigma}^{0+1+2} \otimes su(2) \longrightarrow \Omega_{\Sigma}^{0+1+2} \otimes su(2)$$

given by

$$\hat{D}_A(\alpha, \beta, \gamma) = (*d_{\hat{A}}\beta, -*d_{\hat{A}}\alpha - d_{\hat{A}}*\gamma, d_{\hat{A}}*\beta).$$

(Recall that $A = \hat{A} \times \Theta$ on the collar.)

Now $\hat{D}_A$ is self-adjoint on the closed manifold Σ with symmetric spectrum ($\hat{D}_A\sigma = -\sigma\hat{D}_A$). Moreover, if $\hat{A}$ is flat (e.g. if A is flat) then the kernel of $\hat{D}_A$ is identified with $H^*(\Sigma; ad\rho_A)$ via the Hodge and DeRham theorems. Use the Spectral theorem to decompose $L^2(\Sigma) = L^2(\Omega_{\Sigma}^* \otimes su(2))$ into

$$L^2(\Sigma) = P_-(A) \oplus \mathcal{H}_A \oplus P_+(A)$$

where $P_{\pm}(A)$ is the span of the positive or negative eigenvectors of $\hat{D}_A$ and $\mathcal{H}_A$ is the kernel of $\hat{D}_A$. So in particular $\mathcal{H}_A \cong H^*(\Sigma; ad\rho_A)$. Now $\sigma^2 = -1$ and σ interchanges $P_{\pm}(A)$, and leaves $\mathcal{H}_A$ invariant. Thus σ induces a complex structure on $\mathcal{H}_A$ and, using the L^2 inner product, a symplectic structure via

$$\{x, y\} = \langle x, \sigma(y) \rangle.$$

Since σ depends on the metric, an easy computation shows that the symplectic inner product is metric independent, and in fact coincides with the cup product $\cdot$ on cohomology using $\mathcal{H}_A = H^*(\Sigma; ad\rho_A)$.

Now let $\alpha_t : I \longrightarrow \text{hom}(\pi_1 X, SU(2))$ be an analytic path of representations. Choose an analytic path of flat connections A_t with holonomy α_t in cylindrical form. We now must make the assumption:

Assumption: The dimension of $\mathcal{H}_t$ is independent of t .

This is the analogue of the assumption that the path α_t does not pass through a corner of the pillowcase. More generally it says that the restriction of α_t to the boundary stays in one stratum of $\chi(\Sigma)$. We are *not* assuming that α_t itself stays in one stratum, in fact the entire point of this approach is to analyze what happens as one crosses through singularities. (There exist approaches which suggests how to drop this assumption; as far as I know there is no coherent method of organizing the information as one moves across a singular stratum.)

With this assumption, we can define subbundles of $L^2(\Sigma) \times I$ with fiber over $t \in I$:

$$P_+(t) = \text{span}\{\phi \mid \hat{D}_t\phi = \mu\phi, \mu > 0\}$$

$$\mathcal{H}_t = \ker \hat{D}_t$$

and

$$P_-(t) = \text{span}\{\phi \mid \hat{D}_t \phi = \mu \phi, \mu < 0\},$$

so that $L^2(\Sigma) \times I \longrightarrow I = P_+ \oplus \mathcal{H} \oplus P_- \longrightarrow I$.

The bundle isomorphism σ and the L^2 inner product turns $L^2(\Sigma) \times I$ into a symplectic vector bundle over I and $\mathcal{H}$ into a finite-dimensional symplectic subbundle. (More generally, one obtains a bundle over any parameter space T such that $\dim \mathcal{H}_t$ is constant for $t \in T$.)

The bundle $\mathcal{H}$ is an *analytic* subbundle of $L^2(\Sigma) \times I$; this follows from the fact that the path $\hat{A}_t$ is an analytic path of connections on Σ (and Σ is closed). Choose an analytic *Lagrangian* subbundle $\mathcal{L} \subset \mathcal{H}$. Then the boundary conditions we will use are $P_+ \oplus \mathcal{L}$. Let $\mathcal{D}_t$ mean the operator D_t on X with these boundary conditions. Then it follows from [APS], [Y2], [MW], and many others that $\mathcal{D}_t$ is a self-adjoint operator with discrete spectrum.

Notice that the boundary conditions are varying with t . Applying analytic perturbation theory is not as simple, since the domain of the operators is changing with t . In [KK5] we prove the technical result:

3.13 Theorem.

$\mathcal{D}_t$ is an analytic path of self-adjoint operators. In particular the eigenvalues and eigenvectors vary analytically.

Thus we are poised to apply the same analysis that we did in the closed case to obtain Theorem 3.11. Unfortunately, one more problem creeps in: in the analytic expression for an eigenvector

$$\phi(t) = \sum_{i=0}^{\infty} \phi_i t^i$$

the coefficients ϕ_i need not satisfy the boundary conditions that each $\phi(t)$ does. To make a long story short, this problem is compensated by “stretching the neck”, i.e. replacing X by the manifold

$$X(R) = X \cup (\Sigma \times [0, R])$$

and letting $R \longrightarrow \infty$. The corresponding theorem is weaker, and we state it precisely now. First some notation: the superscript R refers to objects on $X(R)$. The form B_1 is defined to be

$$\tilde{H}^1(X; ad\alpha_0) \times \tilde{H}^1(X; ad\alpha_0) \longrightarrow \mathbf{R}, (x, y) \mapsto -[a, x] \cdot y,$$

where $\tilde{H}^1(X; ad\alpha_0)$ means the image of the relative cohomology in the absolute cohomology. There is also a larger form $\hat{B}_1$ with the same signature as B_1 whose signature is a homotopy invariant. See [KK5] for details; the form B_1 is the “non-hyperbolic part” of $\hat{B}_1$.

3.14 Theorem.: *Given any $\epsilon > 0$, there exists an $R_\epsilon > 0$ so that for all $R > R_\epsilon$, there is a 1-1 correspondence between the eigenvalues of $\hat{B}_1$ and the first derivatives of the eigenvalues $\lambda_i(t)$ of $\mathcal{D}_t^R$ passing through 0 at $t = 0$, denoted by $\tau_i(\hat{B}_1) \leftrightarrow \dot{\lambda}_i^R(0)$, so that*

$$|\tau_i(\hat{B}_1) - \dot{\lambda}_i^R(0)| < \epsilon.$$

In particular, if the cohomology of $(\tilde{H}^(X; ad\alpha_0), [a, -])$ is zero, then the signature of B_1 gives the spectral flow of $D_t^R(P_+(t) + L_t)$ through $t = 0$ for $R > R_\epsilon$, where $\epsilon < \frac{1}{2} \inf |\tau_i(\hat{B}_1)|$.*

This is not quite as good a result as Theorem 3.11, since if some of the $\tau_i(\hat{B}_1)$ are zero we cannot tell what the sign of $\dot{\lambda}_i^R$ is. In the case when all the $\tau_i(\hat{B}_1)$ are non-zero then this gives the computation of spectral flow on $X(R)$ for R large enough. It is probably true that if some $\tau_i(\hat{B}_1)$ vanishes, then so does the corresponding $\dot{\lambda}_i^R$, although a complete proof is not yet available.

3.15 An Example

We finish these lecture notes with a description of a computation using this theorem which verifies a conjecture of Jeffrey [Je] relating the computations of the TQFT invariants of torus bundles over S^1 in the two methods (as described earlier in the notes), namely using the TQFT Axioms as constructed in [Wa] and the stationary phase expansion. For details of the computations see [KK4].

Let M be a torus bundle over a circle, and let $\rho_0, \rho_1 \in \chi(M)$ be two non-abelian representations. In this setting Jeffrey’s conjecture is

$$SF(\rho_0, \rho_1) \equiv 0 \pmod{4}.$$

This conjecture is true if and only if the two interpretations of Witten’s invariant are consistent.

We approach this in our by now familiar way. There exists a simple closed curve K in a fiber so that the restrictions of ρ_i to X_K lie on a path ρ_t in $\chi(X_K)$. (In almost all cases, we were able to prove that the image of this path misses the corners of the pillowcase for a suitable choice of K , but this simple question is the only step which is still missing to verify Jeffrey’s conjecture in its entirety. It is purely a question of arithmetic in $SL(2, \mathbf{Z})$.)

Now it turns out that $\chi(X_K)$ is not always a smooth 1-dimensional variety along ρ_t . If it were, we could apply Theorem 3.5 and immediately compute the spectral flow. However, there are some

2-dimensional components of $\chi(X_K)$ which intersect the path ρ_t transversally for some $0 < t < 1$. This corresponds exactly to the fact that the cohomology jumps up, and so the corresponding path of operators $\mathcal{D}_t$ on X_t picks up some kernel at these special values of t . Figure 13 shows a typical example: ρ_t always maps to a straight line in the pillowcase, and the singularities (i.e. the intersection with the two dimensional components) arise exactly when the image of ρ_t in the pillowcase crosses the top or bottom horizontal edges of the pillowcase. This figure only shows part of $\chi(X_K)$ and its image in the pillowcase. The 2-dimensional sheets cutting through the arc ρ_t map to the top or bottom edges of the pillowcase (i.e. they send λ to ± 1). The picture makes it plain that $\chi(X_K)$ is not a smooth 1-dimensional variety along this arc.

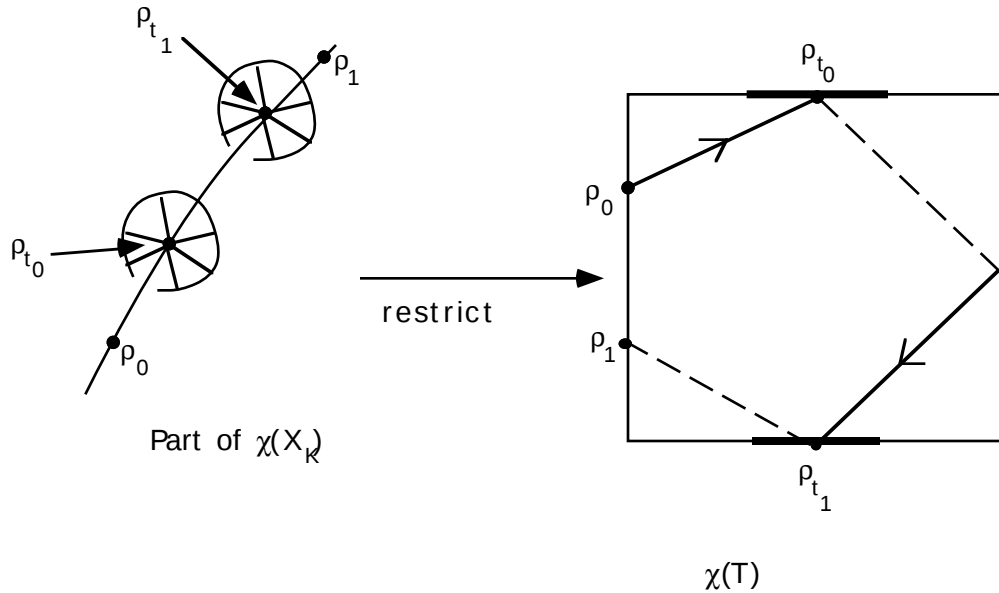


Figure 13.

Now the correct generalization of theorem 3.5 in this situation says that if τ_t is the path in the pillowcase corresponding to the restriction of ρ_t , then

$$(*) \quad SF(\rho_0, \rho_1) = s_{(N,T)}(\tau) + \gamma(\tau) + SF(\mathcal{D}_t) \pmod{8}.$$

The curve τ is a straight line, and is transverse to the vertical line field $\mathcal{L}$ on the pillowcase. Therefore the Maslov index term $\gamma(\tau)$ vanishes. We know how to compute $s_{(N,T)}(\tau)$: this just picks up a ± 2 whenever τ_t crosses the top or bottom horizontal edges of the pillowcase, and a 4 whenever τ_t crosses the left vertical edge. Since $\ker \mathcal{D}_t$ is 2 dimensional whenever τ_t crosses one of the two horizontal edges and zero otherwise, what must be shown is that the two eigenvalues of $\mathcal{D}_t$ crossing through zero at these values of t are crossing with the same sign, if the conjecture is to hold. (If the eigenvalues were crossing with opposite sign, or if one or both had a horizontal

tangency, then the spectral flow of $\mathcal{D}_t$ could be 1 or 0, but then applying the formula (*) we would have an example where $SF(\rho_0, \rho_1)$ is not equal to 0 (mod 4), contradicting the conjecture.)

But Theorem 3.14 tells us how to compute the spectral flow of the family $\mathcal{D}_t$ (at least if the derived cohomology vanishes, which it does in this case). A long but elementary calculation in group cohomology shows that the bilinear form B_1 is definite, and so the first derivatives of the two eigenvalues passing through zero are non-zero. This finishes the proof.

4 Conclusions, Loose Ends

I hope that these lectures have illustrated the point that gauge theory information about 3-manifolds can be obtained by a careful analysis of their character varieties, especially in the context of Dehn surgery, where pictures are possible.

We list some questions which came up in these lectures which need answering.

1. Are all Chern-Simons invariants of representations of closed 3-manifolds rational? See [KK2] and [A] for evidence.
2. An affirmative answer to the following question would make the theory even more useful: Given a closed manifold M and $\rho_0, \rho_1 \in \chi(M)$, does there exist a knot or a link K in M so that the restrictions of ρ_0 and ρ_1 to the exterior X_K lie on a path component of $\chi(X_K)$? The answer is probably no, but under what conditions is this possible?
3. Describe the differentials in Instanton homology in terms of the pillowcase for Dehn surgeries. This is a special case of what I have heard called the Atiyah conjecture. This was presumably answered by Floer in the construction of his exact triangle.
4. Compute the spectral flow along a path which passes through a corner of the pillowcase. This vague question could be answered e.g. by finding the proper generalization of the characteristic cohomology class $s_{(N,T)}$ (perhaps in some generalization of cohomology?).
5. Define the characteristic cohomology class for any 3-manifold pair (X, Σ) and any compact Lie group. (Hint: Answer question 4 first.) Then show that this gives a “torsion” TQFT whose vector spaces associate to Σ some sort of cohomology of $\chi(\Sigma)$ and to X the characteristic class $s_{(X,\Sigma)}$.
6. Referring to example 3.15 (an easy (?) question for number theorists): Prove that given any two representations $\rho_0, \rho_1 \in \chi(M)$ when M is a torus bundle over S^1 , there exists a knot $K \subset M$ which lies in a fiber and a path $\rho_t \in \chi(X_K)$ from $\rho_0|_{X_K}$ to $\rho_1|_{X_K}$ whose image in the pillowcase avoids the corners. (See [KK4] for a reduction of this question to a question about matrices in $SL(2, \mathbf{Z})$.)

7. Calculate $\chi(X)$ for a bunch of hyperbolic knot complements. What can be said about the $SU(2)$ character varieties of hyperbolic knot complements? In general, tables of character varieties of knot exteriors and the image $\chi(X_K) \longrightarrow \chi(T)$ are needed.
8. Does every 3-manifold with non-trivial fundamental group admit a non-trivial representation to $SU(2)$? Does every 3-manifold with non-abelian fundamental group admit a non-abelian representation to $SU(2)$?

References

- [APS] M. Atiyah, V. Patodi, & I. Singer, *Spectral Asymmetry and Riemannian Geometry I,II,III* Math. Proc. Camb. Phil. Soc. **77,78,79** (1975).
- [A] D. Auckly, *Computing Secondary and Spectral Invariants*, Thesis, U. Michigan, 1991.
- [Bo] H. Boden, *Representations of Orbifold Groups and Parabolic Bundles*, Comm. Math. Helv. **66**, (1991), 389-447.
- [B] G. Burde, *$SU(2)$ Representation Spaces for 2-bridge Knot groups* Math. Ann. **288** (1990), 103-119.
- [CLM] S. Cappell, R. Lee, & E. Miller, preprint.
- [CCGLP] D. Cooper, M. Culler, H. Gillet, D. Long, and P. Shalen, *Plane Curves Associated to Character Varieties of 3-Manifolds*, preprint.
- [DK] S. Donaldson and P. Kronheimer, *The Geometry of 4-Manifolds*, Clarendon Press, Oxford (1990).
- [Fr] C. Frohman, *Unitary Representations of Knot Groups*, to appear in Topology.
- [F] A. Floer, *An Instanton invariant for 3 manifolds*, Comm. Math. Phys., **118** (1989), 215–240.
- [FK] C. Frohman and E. Klassen, *Deforming representations of knot groups in $SU(2)$* , ????
- [FKK] B. Fine, P. Kirk, and E. Klassen, *An analytic splitting of the Holonomy Map on Flat Connections*, preprint 1993.
- [FS] R. Fintushel and R. Stern, *Instanton Homology of Seifert-Fibered Homology Spheres*, Proc. Lond. Math. Soc., **61** (1990), 109–137.
- [J] L. Jeffrey, *Chern-Simons-Witten Invariants of Lens spaces and Torus Bundles, and the Semi-Classical Approximation*, Comm. Math. Phys. (1992).
- [GM1] W. Goldman and J. Millson, *The Deformation Theory of Representations of Fundamental Groups of Compact Kahler Manifolds*, Publ. Math. I.H.E.S., vol 67 (1988), 43-96.
- [H] M. Heusener, *$SO(3)$ Representation curves for 2-Bridge Knot Groups*, preprint (1992).
- [He] C. Herald, *Legendrian Cobordism and Chern-Simons Theory on 3-Manifolds With Boundary*, preprint (1993).
- [JS] W. Jaco and P. Shalen
- [Jo] K. Johannson
- [Je] L. Jeffrey *Chern-Simons-Witten Invariants of Lens spaces and Torus Bundles, and the Semi-Classical Approximation*, Comm. Math. Phys. (1992).
- [Ka] T. Kato *Perturbation Theory for Linear Operators* second edition, *Grundlehren der mathematischen Wissenschaft* no. 132, Springer-Verlag.

- [K1] E. Klassen *Representation Spaces of Knot groups in $SU(2)$* Trans A.M.S. **326** (1991), 795-828.
- [K2] E. Klassen, *Representations in $SU(2)$ of the fundamental group of the Whitehead link and of Doubled Knots*, Forum Math. **5**, (1993), 93-109.
- [KK1] P. Kirk and E. Klassen, *Representation spaces of Seifert-Fibered homology Spheres*, Topology, **30** (1991), 77-96.
- [KK2] P. Kirk and E. Klassen, *Chern-Simons Invariants and representation spaces of Knot groups*, Math. Ann. **287** (1990), 343-367.
- [KK3] P. Kirk and E. Klassen, *Chern-Simons Invariants of 3-Manifolds Decomposed along Tori and the Circle Bundle Over the representation Space of T^2* , Comm. Math. Phys. **153** (1993), 521-557.
- [KK4] P. Kirk and E. Klassen *Computing Spectral Flow via Cup Products*, preprint (1992).
- [KK5] P. Kirk and E. Klassen, preprint (1993).
- [KKR] P. Kirk, E. Klassen, & D. Ruberman, *Splitting the Spectral Flow and the Alexander Matrix*. preprint.
- [M] J. Milnor *A Unique Factorization Theorem for 3-Manifolds*, American J. of Math. **84** (1962), 1-7.
- [MW] T. Mrowka and K. Walker,
- [RT] Reshetikin and V. Turaev,
- [RSW] T.R. Ramadas, I. Singer, & J. Weitsman *Some Comments on Chern-Simons Gauge Theory*. Comm. Math. Phys. **126** (1989) 409-420
- [SS] A. Stipsicz and Z. Szabo, *Floer Homology of Certain Algebraic Links*, preprint, 1992.
- [T] C. Taubes, *Gauge theory and Casson's invariant*, J. Diff. Geo. **31** (1990), 547-599.
- [W] E. Witten *Quantum Field Theory and the Jones Polynomial*, Comm. Math.
- [Wa] K. Walker, preprint.
- [Y1] T. Yoshida *The η -invariant of Hyperbolic 3-manifolds*. Invent. Math. **81** (1985) 473-514.
- [Y2] T. Yoshida *Floer homology and splittings of manifolds* Ann. Math. **134** (1991), 277-324.
- [Y3] T. Yoshida *A computational algorithm of spectral flow for Floer homology*, preprint (1990).

Indiana University
Bloomington, Indiana.