Remarks on the cohomology of groups

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1. This is a sequel to two recent papers [3a, b] where I proved some theorems on the deformation of certain types of discrete subgroups of Lie groups; the method consisted in proving (in [3a]) a general result on the small deformations of such groups, and then combining this with the determination of their infinitesimal deformations, as was (in substance) done in [3b]. I have now noticed that, by using an elementary lemma of a general nature, one may deduce the results of [3b] more directly from the knowledge of the infinitesimal deformations, or, what amounts to the same, of the relevant cohomology groups. The first purpose of the present note is to indicate briefly how this can be done.

For simplicity, the lemma in question will be formulated for analytic manifolds, since the real-analytic case is alone relevant to the above-mentioned problems; the lemma is actually valid, not only for analytic manifolds over any complete valued field, but also for manifolds of class C^r (with any $r \geqq 1$) over the reals, and for non-singular algebraic varieties in the sense of abstract algebraic geometry. By a morphism, we understand a mapping of a manifold into a manifold, of the type indicated by the case under consideration (analytic, or of class C^r, or, in the case of algebraic varieties, any everywhere defined mapping in the sense of algebraic geometry).

LEMMA 1. *Let U and V be analytic manifolds and $(W_\iota)_{\iota \in I}$ a family of analytic manifolds; let f be a morphism of U into V, and, for each ι, let F_ι be a morphism of V into W_ι such that $F_\iota \circ f$ is a constant mapping with the constant value c_ι; and put $X = \bigcap_\iota F_\iota^{-1}(c_\iota)$. Let a be a point of U, and $b = f(a)$ its image in V by f; let A, B be the tangent vector-spaces to U at a, and to V at b, respectively; call M the image of A in B under the tangent linear mapping to f at a. Also, for each ι, call L_ι the kernel of the tangent linear mapping to F_ι at b, and put $L = \bigcap_\iota L_\iota$. Assume now that $L = M$; then there is an open neighborhood V_0 of b in V such that, if V' is any open neighborhood of b in V_0, $X \cap V'$ is a submanifold of V' and coincides with the image $f(U) \cap V'$ of $f^{-1}(V')$ under f.*

In some open neighborhood of a in U, take a submanifold U_1 of U with the same dimension as M, containing a, and transversal at a to the kernel of the tangent linear mapping to f at a; then, if A_1 is the tangent linear variety to U_1 at a and if f_1 is the mapping of U_1 into V induced by f, the tangent linear mapping to f_1 at a is an isomorphism of A_1 onto M. On the other hand, for each

ι, take local coordinates $x_\iota^{(i)}$ in a neighborhood of c_ι, with the value 0 at c_ι; then each function $x_\iota^{(i)} \circ F_\iota$ is defined in a neighborhood of b on V; and, among these functions, we can choose finitely many functions φ_ν whose differentials at b are linearly independent, and such that L is defined by the equations $d\varphi_\nu = 0$. We can now choose an open neighborhood V'_0 of b in V where all the φ_ν are everywhere defined, and where their common zeros make up a manifold Y; L is then the tangent linear variety to Y at b, and $X \cap V'_0$ is contained in Y. As f maps U into X, this implies that f maps $f^{-1}(V'_0)$ into Y, and that f_1 maps $f_1^{-1}(V'_0)$ into Y. In the analytic case now under consideration, the assumption $L = M$ implies then that there is a neighborhood of a in U_1 which f_1 maps isomorphically onto a neighborhood of b in Y; the same would be true in the case of C^r-manifolds. In the case of algebraic varieties, the assumption $L = M$ implies that a is an isolated point of $f_1^{-1}(b)$, and hence, by a theorem of Chevalley (re-stated and proved as Lemma 1 in M. Rosenlicht, Trans. A.M.S. 101 (1961), p. 212), that the set-theoretic image under f_1 of any neighborhood of a in U_1 is a neighborhood of b in Y. Thus, in any case, the image of U_1 under f_1 contains an open neighborhood of b in Y, which we may write as $Y \cap V_0$, where V_0 is an open neighborhood of b in V'_0. Let now V' be any open neighborhood of b in V_0; then the image of $f^{-1}(V')$ under f is contained in $X \cap V'$, which is contained in $Y \cap V'$; on the other hand, it contains the image of $f_1^{-1}(V')$ under f_1, which contains $Y \cap V'$; therefore it is the same as $Y \cap V'$ and the same as $X \cap V'$. This completes the proof.

2. Now let ρ be a representation of a group Γ, i.e., a homomorphism of Γ into the automorphism group of a finite-dimensional vector-space V over a field K. Let z be a mapping of Γ into V; for each $\gamma \in \Gamma$, call $\rho'(\gamma)$ the automorphism of the affine space underlying V which is given by $x \to \rho(\gamma)x + z(\gamma)$. Then ρ' is a homomorphism of Γ into the group of automorphisms of that affine space if and only if z satisfies the relation

$$(1) \qquad z(\gamma\gamma') = z(\gamma) + \rho(\gamma)z(\gamma') ,$$

for all γ, γ' in Γ; one expresses this by saying that z is a 1-cocycle (we shall say more briefly a cocycle) of Γ in V, when Γ operates on V by $(\gamma, x) \to \rho(\gamma)x$. Let t_a be the translation $x \to x + a$ in V; then $\rho'(\gamma) = t_a\rho(\gamma)t_a^{-1}$ is equivalent to

$$(2) \qquad z(\gamma) = a - \rho(\gamma)a ;$$

if this is so for all γ, z is a coboundary; the quotient of the space of cocycles by the space of coboundaries is the cohomology space $H^1(\Gamma, V)$.

Let $(\gamma_\alpha)_{\alpha \in A}$ be a family of generators for Γ. Let Γ' be the free group generated by a family of generators $(\gamma'_\alpha)_{\alpha \in A}$ indexed by the same set A; let Δ' be the kernel of the homomorphism of Γ' onto Γ which maps γ'_α onto γ_α for every

α. The elements of Δ' can be considered, in the usual manner, as "words" $w(\gamma')$ in the γ'_α; for every such word, we have $w(\gamma) = \varepsilon$, where ε is the neutral element of Γ. If (w_λ) is a family of elements of Δ', such that Δ' is generated by the w_λ and by their transforms under all inner automorphisms of Γ', then one says that Γ is defined by the generators γ_α with the relations $w_\lambda(\gamma) = \varepsilon$; when that is so, a homomorphism r' of Γ' into a group G with the neutral element e has the constant value e on Δ', and therefore determines a homomorphism r of Γ into G, if and only if $r'(w_\lambda) = e$ for all λ. Since such a homomorphism r' is uniquely determined by the $r_\alpha = r'(\gamma'_\alpha)$, and these can be chosen arbitrarily, we may also say that a homomorphism r of Γ into G is uniquely determined by the elements $r_\alpha = r(\gamma_\alpha)$, and that these can be chosen arbitrarily provided they satisfy the relations $w_\lambda(r) = e$. In particular, let again ρ be a representation of Γ, and let z and ρ' be as above. Then we see that z is a cocycle if and only if $w_\lambda(\rho'(\gamma)) = e$ for all λ. In other words, the cocycle z is uniquely determined by the vectors $z_\alpha = z(\gamma_\alpha)$, and these can be chosen arbitrarily provided they satisfy the relations $w_\lambda(\rho'(\gamma)) = e$. Moreover, z is then a coboundary if and only if there is a vector a such that (2) is satisfied whenever one substitutes one of the γ_α for γ.

In order to write in a convenient form the conditions we have just found for the z_α, write each word $w_\lambda(\gamma)$ as the product $\delta_1 \cdots \delta_{n(\lambda)}$ of a sequence of factors, each one of which is either of the form γ_α or of the form γ_α^{-1}; and put, for $1 \leqq i \leqq n(\lambda)$

$$(3)\qquad \begin{aligned} u_{\lambda i} &= \rho(\delta_1 \cdots \delta_{i-1})\, z_\alpha && \text{if } \delta_i = \gamma_\alpha\,, \\ u_{\lambda i} &= -\,\rho(\delta_1 \cdots \delta_i)\, z_\alpha && \text{if } \delta_i = \gamma_\alpha^{-1}\,. \end{aligned}$$

Then it is easily seen that the relations $w_\lambda(\rho'(\gamma)) = e$ for the z_α are equivalent to the following ones:

$$(4)\qquad \textstyle\sum_{i=1}^{n(\lambda)} u_{\lambda i} = 0\,.$$

These are therefore the relations which determine the cocycles of Γ in V; $H^1(\Gamma, V)$ is 0 if and only if every solution of these equations is a coboundary, i.e., of the form $z_\alpha = a - \rho(\gamma_\alpha)a$ for some choice of a.

3. Now assume that Γ is finitely generated and that $(\gamma_\alpha)_{\alpha\in A}$ is a finite set of generators for Γ, indexed by a finite set A. Let R be the set of all homomorphisms of Γ into a group G; if we identify each $r \in R$ with the point (r_α) of $G^{(A)}$, R is the set of the elements of $G^{(A)}$ which satisfy the relations $w_\lambda(r) = e$. We shall consider only the following two cases:

(a) G is a Lie group; then $G^{(A)}$ is also a Lie group, and R is a real-analytic subset of $G^{(A)}$; in this case, we take for K the field $\mathbf{R}$ of real numbers;

(b) G is an algebraic group; then $G^{(A)}$ is also an algebraic group, and R is a closed subset of $G^{(A)}$ in the Zariski topology; in this case, we take for K the universal domain.

In both cases, each w_λ may be considered as a morphism of $G^{(A)}$ into G, and we have $R = \bigcap_\lambda w_\lambda^{-1}(e)$. On the other hand, for a given $r \in R$, consider all the transforms r' of r by the inner automorphisms of G; these are given by $r'_\alpha = xr_\alpha x^{-1}$, with $x \in G$, and the mapping $x \to (xr_\alpha x^{-1})$ is a morphism of G into $G^{(A)}$, which maps G into R. We shall now apply Lemma 1 to these morphisms. This requires the determination, for the present situation, of the spaces denoted by L and by M in Lemma 1.

We identify the tangent vector-space to G at any point of G with the tangent vector-space to G at e, i.e., with the Lie algebra $\mathfrak{g}$ of G, by means of a right-translation. Then G operates on $\mathfrak{g}$ by means of the adjoint group; more precisely, the inner automorphism $x \to sxs^{-1}$ of G induces on the tangent vector-space $\mathfrak{g}$ of G at e the automorphism $X \to \mathrm{Ad}(s)X$.

It is now easily seen that the image M of the tangent vector-space $\mathfrak{g}$ to G at e, under the tangent mapping to the morphism $x \to (xr_\alpha x^{-1})$ at e, consists of the vectors $(Z_\alpha) \in \mathfrak{g}^{(A)}$ of the form $Z_\alpha = X - \mathrm{Ad}(r_\alpha)X$, where X is any vector in $\mathfrak{g}$. On the other hand, write again $w_\lambda(\gamma)$ as $\delta_1 \cdots \delta_{n(\lambda)}$, where the δ_i are as above; then the image of a tangent vector (Z_α) to $G^{(A)}$ at r, under the tangent mapping to the morphism w_λ of $G^{(A)}$ into G, is $\sum_i u_{\lambda i}$, where the $u_{\lambda i}$ are given by (3), provided one substitutes (Z_α) for (z_α) and $\mathrm{Ad}\circ r$ for ρ in (3). Therefore, after this substitution is made, the kernel L_λ of the tangent mapping to w_λ at r is defined by (4); and those equations, taken for all λ, define the linear variety $L = \bigcap_\lambda L_\lambda$. In other words, L is the space of the cocycles of Γ in $\mathfrak{g}$, when Γ operates on $\mathfrak{g}$ by $(\gamma, X) \to \mathrm{Ad}(r(\gamma))X$, while M is the space of coboundaries for the same group; $L = M$ is equivalent to $H^1(\Gamma, \mathfrak{g}) = 0$. In view of Lemma 1, this gives the following theorem:

If $H^1(\Gamma, \mathfrak{g}) = 0$, there is a neighborhood of r in which every element of R is the transform of r by an inner automorphism of G.

As a consequence, in order to prove Theorem 1 of [3b], one has only to verify that, under the assumptions of that theorem, $H^1(\Gamma, \mathfrak{g}) = 0$; in substance, this is precisely what is done in nos. 6–10 of that paper (cf. also [4]).

4. In the cases covered by Theorems 2 and 3 of [3b], $H^1(\Gamma, \mathfrak{g})$ is not 0; nevertheless, these theorems can be proved by a similar method, viz., by applying Lemma 1 to the situation described above in nos. 2–3, and combining this with the information about $H^1(\Gamma, \mathfrak{g})$ contained in [3b] and with the direct determination of this group for the case in which Γ is a discrete subgroup, with compact

quotient-space, of the 3-dimensional hyperbolic group. It seems hardly worthwhile to give a detailed proof along these lines, which would not lead to any new result; but I shall take this opportunity for giving some results about the cohomology of the groups Γ of the last-mentioned type, since this also plays a role in other investigations (cf. e.g., [1] and [2]). As in [1], we consider more generally the discrete subgroups Γ of the hyperbolic group for which the quotient-space has a finite measure. It is known that these groups can be obtained as follows.

To begin with, take a free group Λ with $2g+n$ generators $A_1, \cdots, A_g$, $B_1, \cdots, B_g, C_1, \cdots, C_n$ (where $g \geqq 0$, $n \geqq 0$); call E the neutral element of that group. Put $R_0 = E$, and define elements (or "words") $R_1, \cdots, R_{g+n}$ of that group inductively by the following formulas

$$R_\alpha = R_\alpha(A, B, C) = R_{\alpha-1}A_\alpha B_\alpha A_\alpha^{-1}B_\alpha^{-1} \qquad (1 \leqq \alpha \leqq g)\,, \tag{5}$$

$$R_{g+\nu} = R_{g+\nu}(A, B, C) = R_{g+\nu-1}C_\nu \qquad (1 \leqq \nu \leqq n)\,; \tag{6}$$

actually, for $1 \leqq \alpha \leqq g$, the word R_α contains only the A_β and B_β for $1 \leqq \beta \leqq \alpha$ and could have been written $R_\alpha(A, B)$.

Choosing now n integers $s_\nu \geqq 0$, we take for Γ the group with the $2g+n$ generators $a_1, \cdots, a_g, b_1, \cdots, b_g, c_1, \cdots, c_n$ and the defining relations

$$R_{g+n}(a, b, c) = e\,; \qquad c_\nu^{s_\nu} = e \qquad (1 \leqq \nu \leqq n)\,, \tag{7}$$

where e is the neutral element of Γ; we shall denote by λ the homomorphism of Λ onto Γ which maps $A_\alpha, B_\alpha, C_\nu$ onto $a_\alpha, b_\alpha, c_\nu$, respectively, for $1 \leqq \alpha \leqq g$, $1 \leqq \nu \leqq n$.

5. We shall now define an involutory automorphism of Γ which will be needed in our discussion of the cohomology of Γ. Define elements $A'_\alpha, B'_\alpha, C'_\nu$ of Λ by putting

$$A'_\alpha = F_\alpha(A, B) \quad = R_{\alpha-1}B_\alpha^{-1}R_\alpha^{-1} \qquad (1 \leqq \alpha \leqq g)\,, \tag{8}$$

$$B'_\alpha = G_\alpha(A, B) \quad = R_\alpha A_\alpha^{-1}R_{\alpha-1}^{-1} \qquad (1 \leqq \alpha \leqq g)\,, \tag{9}$$

$$C'_\nu = H_\nu(A, B, C) = R_{g+\nu-1}R_{g+\nu}^{-1} \qquad (1 \leqq \nu \leqq n)\,. \tag{10}$$

Then a trivial induction shows that we have the relation

$$R'_i = R_i(A', B', C') = R_i^{-1} \qquad (1 \leqq i \leqq g+n)\,, \tag{11}$$

which gives at once

$$F_\alpha(A', B') = A_\alpha\,, \qquad G_\alpha(A', B') = B_\alpha \qquad (1 \leqq \alpha \leqq g)\,,$$
$$H_\nu(A', B', C') = C_\nu \qquad (1 = \nu \leqq n)\,;$$

this shows that there is an involutory automorphism Φ of Λ which maps A_α, B_α, C_ν onto $A'_\alpha, B'_\alpha, C'_\nu$, respectively. From (6), (10) and (11), it follows that we have

$$\Phi C_\nu^{s_\nu} = R_{g+\nu-1} C_\nu^{-s_\nu} R_{g+\nu-1}^{-1} , \qquad \Phi(R_{g+n}) = R_{g+n}^{-1} ,$$

which implies that Φ maps the kernel of λ onto itself; therefore the relation $\varphi\circ\lambda = \lambda\circ\Phi$ determines an involutory automorphism φ of Γ. Put $a'_\alpha = \varphi(a_\alpha)$, $b'_\alpha = \varphi(b_\alpha)$, $c'_\nu = \varphi(c_\nu)$.

Assume now that Λ operates on a vector-space V by $(X, x) \to X\cdot x$; then, for any cocyle z of Λ in V, we have

$$(12) \qquad z(R_\alpha) = \sum_{\beta=1}^{\alpha} (A'_\beta - E) R_\beta B_\beta \cdot z(A_\beta) - \sum_{\beta=1}^{\alpha} (B'_\beta - E) R_{\beta-1} A_\beta \cdot z(B_\beta) \qquad (1 \leqq \alpha \leqq g) ,$$

$$(13) \qquad z(R_{g+\nu}) = z(R_g) + \sum_{\mu=1}^{\nu} R_{g+\mu-1}\cdot z(C_\mu) \qquad (1 \leqq \nu \leqq n) ,$$

and also, for every integer $s \geqq 0$:

$$(14) \qquad z(C_\nu^s) = (E + C_\nu + \cdots + C_\nu^{s-1})\cdot z(C_\nu) .$$

In particular, if ρ is a representation of Γ in a vector-space V over a field K, we can apply (12), (13) and (14) to the representation $\rho\circ\lambda$ of Λ in V. On the other hand, the remarks in no. 2 show that a cocycle z of Γ in V is uniquely determined by the vectors $z(a_\alpha)$, $z(b_\alpha)$, $z(c_\nu)$, and that these can be chosen arbitrarily provided they satisfy the relations obtained by writing

$$\bar{z}(R_{g+n}) = 0 , \qquad \bar{z}(C_\nu^{s_\nu}) = 0$$

for the cocycle $\bar{z}$ of Λ in V defined by

$$\bar{z}(A_\alpha) = z(a_\alpha) , \qquad \bar{z}(B_\alpha) = z(b_\alpha) , \qquad \bar{z}(C_\nu) = z(c_\nu) .$$

6. For each ν, call Γ_ν the subgroup of Γ generated by c_ν; it is isomorphic to $\mathbf{Z}$ if $s_\nu = 0$, and cyclic of order s_ν if $s_\nu > 0$. In view of (14), and with the notations which have just been explained, the relation $\bar{z}(C_\nu^{s_\nu}) = 0$ can be written

$$(15) \qquad \sum_{h=0}^{s_\nu-1} \rho(c_\nu^h) z(c_\nu) = 0 ,$$

which expresses that $z(c_\nu)$ determines a cocycle of Γ_ν in V; this is a coboundary if and only if $z(c_\nu)$ is of the form

$$(16) \qquad z(c_\nu) = w_\nu - \rho(c_\nu) w_\nu$$

with $w_\nu \in V$. We shall say that a cocycle z of Γ in V is *parabolic* if, for every ν, it induces a coboundary on Γ_ν, i.e., if $z(c_\nu)$ is of the form (16) for every ν. Every coboundary of Γ in V is of course a parabolic cocycle; we shall write $P^1(\Gamma, V)$ for the quotient of the space of parabolic cocycles by the space of coboundaries of Γ in V. It is easily seen (and, of course, well-known) that every solution of (15) is of the form (16) if s_ν is not a multiple of the characteristic of K; if that is so for every ν, every cocycle of Γ in V is parabolic, and $P^1(\Gamma, V) = H^1(\Gamma, V)$; in particular, this is so if K is of characteristic 0 and none of the s_ν is 0.

From the above remarks, it follows that a parabolic cocycle z is uniquely determined by the vectors $z(a_\alpha)$, $z(b_\alpha)$, w_ν, and that these can be chosen arbitrarily, provided they satisfy the relation which was written above as $\bar{z}(R_{g+n})=0$; this can at once be written in terms of the $z(a_\alpha)$, $z(b_\alpha)$, w_ν by using (12), (13) and (16). In order to write it more conveniently, put

$$(17)\qquad \begin{aligned} u_\alpha &= \rho\big(R_\alpha(a,\, b,\, c)b_\alpha\big)\, z(a_\alpha) && (1 \leqq \alpha \leqq g)\,, \\ v_\alpha &= \rho\big(R_{\alpha-1}(a,\, b,\, c)a_\alpha\big)\, z(b_\alpha) && (1 \leqq \alpha \leqq g)\,, \\ t_\nu &= \rho\big(R_{g+\nu}(a,\, b,\, c)\big)\, w_\nu && (1 \leqq \nu \leqq n)\,. \end{aligned}$$

The relation in question can then be written as follows

$$(18)\qquad \textstyle\sum_\alpha [\rho(a'_\alpha) - 1_V]u_\alpha - \sum_\alpha [\rho(b'_\alpha) - 1_V]v_\alpha + \sum_\nu [\rho(c'_\nu) - 1_V]t_\nu = 0\,,$$

where $1_V = \rho(e)$ is the identity automorphism of V. Call $F(u,\, v,\, t)$ the left-hand side of (18); F is a linear mapping of V^{2g+n} into V. Let d be the dimension of V, and let i' be the codimension in V of the image of V^{2g+n} under F; then the kernel of F, i.e., the space of the solutions $(u,\, v,\, t)$ of (18), has the dimension $D = (2g+n)d - d + i'$. On the other hand, for each ν, call e_ν the rank of the endomorphism $1_V - \rho(c_\nu)$ of V. Then the kernel of the mapping $w_\nu \to z(c_\nu)$ defined by (16) has the dimension $d - e_\nu$; therefore, in the mapping of the space of solutions $(u,\, v,\, t)$ of (18) into the space of parabolic cocycles which is defined by (16) and (17), the dimension of the kernel is $\sum_\nu (d - e_\nu)$. This gives, for the space of parabolic cocycles, the dimension

$$(19)\qquad P = (2g-1)d + i' + \textstyle\sum_\nu e_\nu\,.$$

In order to give a more convenient interpretation for i' than the one given above, let V' be the dual space to V; write $\langle x,\, x'\rangle$ for the bilinear form on $V \times V'$ which defines the duality between V and V'; as usual, denote by ${}^t\rho(s)$, for $s \in \Gamma$, the transpose of $\rho(s)$, i.e., the automorphism of V' defined by

$$\langle \rho(s)x,\, x'\rangle = \langle x,\, {}^t\rho(s)x'\rangle\,.$$

By definition, i' is the dimension of the space of vectors $x' \in V'$ such that

$$\langle F(u,\, v,\, t),\, x'\rangle = 0$$

for all $(u,\, v,\, t)$. This is clearly equivalent to the relations

$${}^t\rho(a'_\alpha)x' = x'\,, \qquad {}^t\rho(b'_\alpha)x' = x'\,, \qquad {}^t\rho(c'_\nu)x' = x'\,.$$

As the a'_α, b'_α, c'_ν make up a set of generators for Γ, this shows that *i' is the dimension of the space of the vectors in V' which are invariant under ${}^t\rho(s)$ for all $s \in \Gamma$.*

On the other hand, *call i the dimension of the space of the vectors in V which are invariant under $\rho(s)$ for all $s \in \Gamma$.* This is the dimension of the kernel of the mapping $a \to z$ of V onto the space of coboundaries of Γ which is defined by (2); therefore that space has the dimension $d - i$, and we get, for the dimension p of the space $P^1(\Gamma, V)$, the formula

$$p = (2g - 2)d + i + i' + \sum_\nu e_\nu .$$

This gives the dimension of the cohomology space $H^1(\Gamma, V)$ in the cases in which all cocycles are parabolic (e.g., if no s_ν is 0 and K is of characteristic 0).

7. As in no. 3, these results can be applied to the study of the space of the representations of Γ into a group G of one of the types introduced there, i.e., a Lie group in case (a) and an algebraic group in case (b). We assume that none of the s_ν is a multiple of the characteristic of the field K (i.e., in case (a), that none of them is 0); then every cocycle of Γ is parabolic; more precisely, as we have already observed in no. 6, the cohomology space $H^1(\Gamma_\nu, V)$, for any representation of Γ_ν in a vector-space over K, is 0. For each ν, call W_ν the set of the elements w of G which satisfy $w^{s_\nu} = e$; the theorem at the end of no. 3 shows that W_ν is an analytic submanifold of G in case (a), and that it is the union of finitely many mutually disjoint non-singular subvarieties of G in case (b); moreover, it shows that the connected component in W_ν of any point w of W_ν consists of the points xwx^{-1} for $x \in G$, and that the tangent linear variety to that component at w is the image M_ν of $\mathfrak{g}$ under the tangent linear mapping to $x \to xwx^{-1}$ at e.

Now let R be the set of the homomorphisms of Γ into G; for each $r \in R$, put $r_\alpha = r(a_\alpha)$, $r'_\alpha = r(b_\alpha)$, $r''_\nu = r(c_\nu)$, and identify r with the point $(r_\alpha, r'_\alpha, r''_\alpha)$ of $G^{2g} \times W$, where $W = \prod_\nu W_\nu$. We can consider R_{g+n} as defining, in an obvious manner, a mapping of $G^{2g} \times W$ into G, and the set R is then nothing else than $R_{g+n}^{-1}(e)$. By the same kind of calculations as those in nos. 3, 5, 6, it is easy to determine the tangent linear mapping to R_{g+n} at any point, or, what amounts to the same, the tangent linear variety S to the graph of R_{g+n} at any point $(r, R_{g+n}(r))$ of that graph, and in particular, at the point (r, e) of this graph for any $r \in R$; if we take again for ρ, as in no. 3, the homomorphism $\mathrm{Ad}\circ r$ of Γ into the group of automorphisms of $\mathfrak{g}$, and if we write again $F(u, v, t)$ for the left-hand side of (18), we find that S is the space of all vectors of the form

$$\big(z(a_\alpha), z(b_\alpha), z(c_\nu), F(u, v, t)\big)$$

when one takes for $(z(a_\alpha), z(b_\alpha))$ all the vectors in $\mathfrak{g}^{2g}$, for (w_ν) all the vectors in $\mathfrak{g}^n$, and for (u, v, t) and $(z(c_\nu))$ the vectors defined in terms of these by (16) and (17). With the notations of no. 6, S has then the dimension $2gd + \sum_\nu e_\nu$, d being the dimension of G. The points of S for which $F(u, v, t) = 0$ are no other than the cocycles of Γ in $\mathfrak{g}$, and make up a subspace of S whose dimension P is given by (19); its codimension in S is d if and only if $i' = 0$. This shows that the graph of R_{g+n} is transversal to $(G^{2g} \times W) \times e$ at (r, e), for a given $r \in R$, if and only if $i' = 0$. Therefore, *when* $i' = 0$, *there is a neighborhood of* r *in which* R *is an analytic submanifold of* $G^{2g} \times W$ *in case* (a), *and a non-singular subvariety of* $G^{2g} \times W$ *in case* (b), *with the dimension* $P = (2g - 1)d + \sum_\nu e_\nu$.

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