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REPRESENTATIONS OF KNOT GROUPS IN SU(2)

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ABSTRACT. This paper is a study of the structure of the space $R(K)$ of representations of classical knot groups into $\mathrm{SU}(2)$. Let $\widehat{R}(K)$ equal the set of conjugacy classes of irreducible representations.

In §I, we interpret the relations in a presentation of the knot group in terms of the geometry of $\mathrm{SU}(2)$; using this technique we calculate $\widehat{R}(K)$ for K equal to the torus knots, twist knots, and the Whitehead link. We also determine a formula for the number of binary dihedral representations of an arbitrary knot group. We prove, using techniques introduced by Culler and Shalen, that if the dimension of $\widehat{R}(K)$ is greater than 1, then the complement in S^3 of a tubular neighborhood of K contains closed, nonboundary parallel, incompressible surfaces. We also show how, for certain nonprime and doubled knots, $\widehat{R}(K)$ has dimension greater than one.

In §II, we calculate the Zariski tangent space, $T_\rho(R(K))$, for an arbitrary knot K, at a reducible representation ρ, using a technique due to Weil. We prove that for all but a finite number of the reducible representations, $\dim T_\rho(R(K)) = 3$. These nonexceptional representations possess neighborhoods in $R(K)$ containing only reducible representations. At the exceptional representations, which correspond to real roots of the Alexander polynomial, $\dim T_\rho(R(K)) = 3 + 2k$ for a positive integer k. In those examples analyzed in this paper, these exceptional representations can be expressed as limits of arcs of irreducible representations. We also give an interpretation of these "extra" tangent vectors as representations in the group of Euclidean isometries of the plane.

0. INTRODUCTION

This paper is a study of the space $R(K)$ of representations of the fundamental group of the complement of a knot K into the Lie group $\mathrm{SU}(2)$. The major results are as follows: (1) a calculation of the topological type of $R(K)$ for K a torus knot, a twist knot, or the Whitehead link, (2) a determination of the Zariski tangent space to $R(K)$ at a reducible representation for an arbitrary knot K, and (3) a proposition and some examples relating the dimension of $R(K)$ to incompressible surfaces in the complement of K.

While the idea of representing 3-manifold groups in $\mathrm{SU}(2)$ is a fairly recent one, the representation of these groups in other groups has a substantial history. We will not attempt to chronicle this history in detail, but will comment here on

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those developments which we consider most relevant to this paper: In *A quick trip through knot theory* [F, pp. 160–163], Fox analyzes cyclic and metacyclic representations of knot groups. Weil in [W], did important work on calculating the Zariski tangent space to a variety of representations into a Lie group. Riley [Ri1, Proposition 4] has exhibited specific arcs of representations of 2-bridge knot groups into $\mathrm{SU}(2)$. There is some overlap between Riley's work and this thesis, since the twist knots and some of the torus knots (the ones of type $(2, q)$) considered here are 2-bridge knots. Using methods similar to those of Riley, Gerhard Burde [B] has computed $\widehat{R}(K)$ for certain 2-bridge knots K, including the twist knots. In particular, he independently proved Theorem 3. Culler and Shalen, in [CS], develop a beautiful relationship between curves of representations in $\mathrm{SL}(2, \mathbb{C})$ and incompressible surfaces. Finally, Andrew Casson has constructed an important new invariant of homology 3-spheres using their representation spaces in $\mathrm{SU}(2)$ (see [AM]).

Organization. Let $K \subset S^3$ be a knot and define $R(K)$ to be the space of representations $\mathrm{Hom}(\pi_1(S^3 - K), \mathrm{SU}(2))$. Let $\widehat{R}(K)$ be the set of conjugacy classes of irreducible representations. (These terms will be discussed more fully in §I.A.) This paper falls naturally into two sections.

§I is devoted to a geometric understanding of the structure of $\widehat{R}(K)$. In §I.A, some useful facts about the geometry of $\mathrm{SU}(2)$ are stated, and the basic objects of study are defined. In §I.B the topological type of $\widehat{R}(K)$ is computed for K equal to a torus knot, and an application is given concerning Casson's invariant for homology spheres constructed by $1/n$ surgery on the $(3, 4)$-torus knot. In §I.C, the topological type of $\widehat{R}(K)$ is computed for K an m-twist knot. To perform this computation, a basic method is introduced whereby $\widehat{R}(K)$ is identified with a certain set of geometric immersions of a polygon (which depends on the projection of the knot K) in S^2. In §I.D (Theorem 6), the topological type of $\widehat{R}(\mathrm{WL})$ is computed, where WL is the Whitehead link. This computation is of use in computing $\widehat{R}$(doubled knots), an endeavor which is touched on in §I.F, and which will be explored more fully in a future paper (see [K]). In §I.E (Theorem 10), a formula is derived for the number of conjugacy classes of binary dihedral representations (which comprise a special class of irreducible representations into $\mathrm{SU}(2)$) of an arbitrary knot group. In §I.F (Proposition 15) we show, using results of Culler and Shalen, that if $\widehat{R}(K)$ contains a component of dimension greater than one, then $S^3 - \mathrm{int}(N(K))$ contains nonboundary parallel, closed, incompressible surfaces. We also show, by analyzing composite knots and satellite knots, that incompressible tori in a knot complement often do, in fact, lead to components of $\widehat{R}(K)$ of dimension greater than one (Propositions 13 and 14).

§II is devoted to the computation and interpretation of the Zariski tangent space to $R(K)$, $T_\rho(R(K))$, at a reducible representation ρ. In §II.A, we show how to view $R(K)$ as an algebraic set, and prove the basic lemma (Proposition 18, stated by Weil in [W, p. 151]) which makes it possible to calculate

$T_\rho(R(K))$. In §II.B, we apply these techniques to compute $R_\rho(R(K))$, where ρ is reducible. As a result we find (Theorem 19) that, except for a finite number (up to conjugacy) of exceptions, the reducible representations are 3-manifold points of $R(K)$. The finite number of exceptional reducible representations, which correspond to roots of the Alexander polynomial of K, have higher dimensional tangent spaces. In the case of the knots analyzed in §I, these extra, "irreducible" tangent vectors correspond to actual irreducible deformations of these reducible representations. We also show (Proposition 20) that these "irreducible" tangent vectors can be interpreted as nonabelian representations of the knot group into the group of isometries of the Euclidean plane. These Euclidean representations have been analyzed by De Rham [D].

I. Geometric computation of representation spaces

I.A. Preliminaries on SU(2) representations. We will begin by defining the two Lie groups, SU(2) and S^3, with which this paper is most concerned:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\},$$

$$S^3 = \{\text{quaternions } q \text{ such that } |q| = 1\}.$$

These two groups are isomorphic via the map given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + b\mathbf{j}.$$

Using this isomorphism we will refer to these groups pretty much interchangeably, though we will more often use quaternion notation.

Let Γ be a finitely presented group with presentation

$$\Gamma = \{x_1, x_2, \dots, x_n : w_1, w_2, \dots, w_m\},$$

where the relators w_i are words in the generators x_j. Define

$$R(\Gamma) = \mathrm{Hom}(\Gamma, S^3),$$

and define

$$S(\Gamma) = \{\rho \in R(\Gamma) | \operatorname{im}(\rho) \text{ is abelian}\}.$$

These abelian representations are often referred to as "reducible" because a representation of any group in SU(2) is reducible if and only if it has abelian image. We give S^3 its usual topology and Γ the discrete topology; we then give $R(\Gamma)$ the compact-open topology.

$S^3/\{\pm 1\} \cong \mathrm{SO}(3)$ acts on $R(\Gamma)$ by conjugation:

$$(q \cdot \rho)(x) = q\rho(x)q^{-1},$$

where $q \in S^3$, $x \in \Gamma$, and $\rho \in R(\Gamma)$. It is easily verified (see, for example, [AM]), that this action is free when restricted to $R(\Gamma) - S(\Gamma)$. Hence, define

$$\widehat{R}(\Gamma) = (R(\Gamma) - S(\Gamma))/\text{action by } \mathrm{SO}(3).$$

Using the above presentation for Γ, we can embed $R(\Gamma)$ in $(S^3)^n$ via the map

$$f(\rho) = (\rho(x_1), \dots, \rho(x_n)).$$

The map f is injective because the x_i generate Γ. We now characterize the image of f. Let $(\sigma_1, \sigma_2, \dots, \sigma_n)$ be an element of $(S^3)^n$. If we substitute σ_i for each x_i in the word w_j, then we can view each w_j as a map from $(S^3)^n$ to S^3. It follows that

$$\operatorname{im}(f) = \bigcap \{w_i^{-1}(1) : i = 1, \dots, m\}.$$

The w_i are polynomials in the ambient coordinates, so $\operatorname{im}(f)$ is an algebraic subset of $(S^3)^n$. We will henceforth identify $R(\Gamma)$ with $\operatorname{im}(f)$. In [LM], Lubotzky and Magid show how $R(\Gamma)$ may be given the structure of an affine algebraic scheme which is independent of the particular presentation of Γ.

Let $K \subset S^3$ be a knot. To simplify notation, we write $R(K)$, $S(K)$, and $\widehat{R}(K)$ instead of $R(\pi_1(S^3 - K))$, $S(\pi_1(S^3 - K))$, and $\widehat{R}(\pi_1(S^3 - K))$, respectively. $R(K)$ and $\widehat{R}(K)$ will be the main objects of study in this thesis.

Geometry of S^3. We now state (mostly without proof—see [S] for some of these computations) some basic geometric facts about the group S^3. Let $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in S^3$. Define $\operatorname{Re}(q) = a$. Assume $q \neq \pm 1$. We can express q in the form

$$q = \cos\theta + \sin\theta(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}),$$

where $a^2 + b^2 + c^2 = 1$. If we fix a, b, and c, while allowing $\theta \in \mathbb{R}$ to vary, we obtain a subgroup S^1_q of S^3 which is isomorphic to $S^1 \subset \mathbb{C}$. Indeed, $S^1_{\pm i} = S^1 \subset \mathbb{C}$. Every $q \neq \pm 1$ is contained in a unique such circle subgroup. These subgroups are just the geodesics through 1 in S^3. Any two of these circle subgroups are conjugate to each other in S^3.

Consider the map $\varphi : S^3 \to [-1, 1]$ given by $\varphi(q) = \operatorname{Re}(q)$. If $t \neq \pm 1$, then $\varphi^{-1}(t)$ is a 2-sphere. We will use the notation

$$\Sigma_t = \varphi^{-1}(t).$$

Note that if $q \neq \pm 1 \in S^3$ and $t \in (-1, 1)$, then S^1_q meets Σ_t in two points, and they are orthogonal at these points.

We now consider the geometric effect of conjugation in S^3. Define $C_q : S^3 \to S^3$ by $C_q(w) = qwq^{-1}$. It is easily verified that $C_q(\Sigma_t) = \Sigma_t$ for $-1 \leq t \leq 1$. In fact, C_q acts on Σ_t as a rotation by angle 2θ about the two antipodal points where Σ_t is pierced by S^1_q. (We are writing

$$q = \cos\theta + \sin\theta(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}),$$

where $0 \leq \theta \leq \pi$.) The direction of this rotation is determined by which of the two intersection points is closer to q in S^3. It is a right-handed rotation about

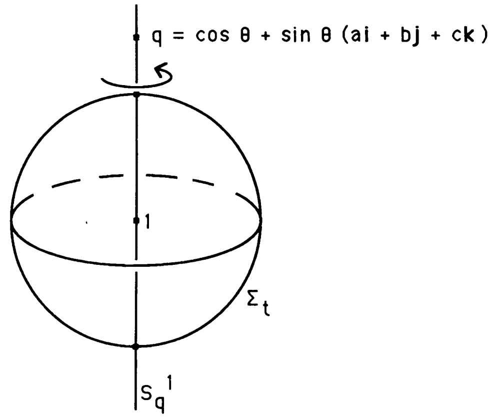


FIGURE 1. Conjugation by q induces a rotation by an angle of 2θ on Σ_t

this closer intersection point. It follows that the Σ_t are precisely the conjugacy classes of S^3. Figure 1 illustrates these relationships in S^3.

I.B. Torus knots. Let (r, s) be any pair of positive, relatively prime integers. Let $K_{r,s}$ denote the (r, s)-torus knot in S^3.

Theorem 1. *$\widehat{R}(K_{r,s})$ is the disjoint union of $(r-1)(s-1)/2$ open arcs.*

Proof. In this proof, we assume that r and s are odd. The case of r or s even is handled by the same method. $\Gamma = \pi_1(S^3 - K_{r,s})$ has a presentation of the form

$$\Gamma = \{x, y | x^r = y^s\}.$$

Since we are considering classes of nonabelian representations, we consider only representations ρ such that $[\rho(x), \rho(y)] \neq 1$. It follows that $\rho(x) \neq \pm 1 \neq \rho(y)$, and that $\rho(x)$ and $\rho(y)$ lie in distinct circle subgroups $S^1_{\rho(x)}$ and $S^1_{\rho(y)}$, which we denote simply by S^1_x and S^1_y. We know that

$$(*) \qquad \rho(x)^r = \rho(y)^s \in S^1_x \cap S^1_y = \{\pm 1\}.$$

Consider the case $\rho(x)^r = \rho(y)^s = +1$. It follows that

$$\rho(x) \in \{q | q^r = 1, q \neq 1\} = \bigcup \{\Sigma_{t_x} : t_x \in A\},$$

where $A = \{\cos(2\pi/r), \cos(4\pi/r), \ldots, \cos((r-1)\pi/r)\}$. This gives $(r-1)/2$ possible conjugacy classes for $\rho(x)$. Since s is odd as well, there are $(s-1)/2$ possible conjugacy classes for $\rho(y)$. When we choose conjugacy classes Σ_{t_x} for $\rho(x)$ and Σ_{t_y} for $\rho(y)$ we constrain these images to lie in two concentric 2-spheres centered at 1 in S^3. (They are concentric with respect to geodesic radii emanating from $1 \in S^3$.) The only remaining conjugacy invariant for the pair $(\rho(x), \rho(y))$ is the angle α between the shortest geodesic from 1 to $\rho(x)$

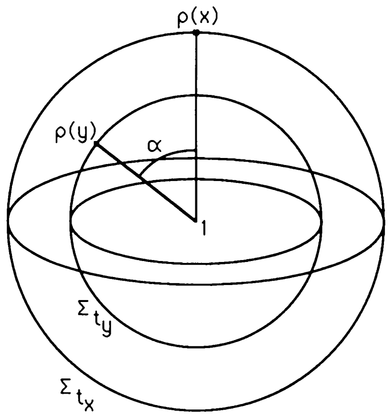


FIGURE 2. The angular invariant α for pairs $(\rho(x), \rho(y))$

and the shortest geodesic from 1 to $\rho(y)$. This is because conjugation of ρ induces the "same" rotation on Σ_{t_x} and Σ_{t_y}.

If $\alpha = 0$ or $\alpha = \pi$, then ρ is abelian. This leaves an open arc (corresponding to $0 < \alpha < \pi$) of conjugacy classes of nonabelian representations for each pair of conjugacy classes $(\Sigma_{t_x}, \Sigma_{t_y})$ for $(\rho(x), \rho(y))$. It follows that there are $(r-1)(s-1)/4$ open arcs of classes of representations for which $\rho(x)^r = \rho(y)^s = +1$. A similar computation yields another $(r-1)(s-1)/4$ open arcs of classes of representations satisfying $\rho(x)^r = \rho(y)^s = -1$. The theorem follows. □

Let M_n be the homology 3-sphere obtained by a Dehn surgery of type $1/n$ on $K_{3,4}$. (For conventions regarding Dehn surgery on knots, see Rolfsen [Ro, pp. 258–259].) Let

$$\widehat{R}(M_n) = \mathrm{Hom}(\pi_1(M_n), S^3)/\text{conjugacy in } S^3.$$

Proposition 2. *The cardinality of* $\widehat{R}(M_n)$ *is equal to* $|10n|$.

Note. Casson's invariant for homology 3-spheres, $\lambda(M^3)$, is defined to be $1/2$ the algebraic intersection number of two manifolds whose geometric intersection is $\widehat{R}(M^3)$. Hence, for a given homology 3-sphere M^3 it is natural to ask the question: does $|\lambda(M^3)| = \mathrm{card}(\widehat{R}(M^3))/2$? The above proposition answers this question affirmatively for those homology 3-spheres obtained by Dehn surgery on $K_{3,4}$, since $\lambda(M_n) = n\lambda'(K_{3,4}) = n\Delta''_{K_{3,4}}(1) = 5n$. ($\Delta_{K_{3,4}}(t)$ is the Alexander polynomial normalized so that the coefficient of t^k is the same as the coefficient of t^{-k} for each value of k. See [AM] for details of this computation.) In [BN], Boyer and Nicas carry out this calculation for homology 3-spheres obtained from surgery on $(2, q)$-torus knots, and find agreement there, also, between $|\lambda(M^3)|$ and $\mathrm{card}(\widehat{R}(M^3))/2$.

Added in proof. The author has learned that Fintushel and Stern have recently shown that for all Seifert-fibered homology spheres M^3 with three exceptional fibers,

$$\lambda(M^3) = \tfrac{1}{2}\operatorname{card}(\widehat{R}(M^3)).$$

This class of homology spheres includes those obtained by surgery on torus knots. Their proof uses Floer's recent reformulation of Casson's λ-invariant in terms of gauge theory.

Proof of Proposition 2. In this proof, we shall assume that $n > 0$. The case $n < 0$ is handled in the same way, and is omitted.

The group $\Gamma = \pi_1(S^3 - K_{3,4})$ has presentation

$$\Gamma = \{x, y : x^3 = y^4\}.$$

In Γ, a meridian can be represented by $\mu = xy^{-1}$, and a longitude by $\lambda = x^3(xy^{-1})^{-12}$. To obtain $\pi_1(M_n)$, we adjoin the relation $\mu\lambda^n = 1$ to our presentation of Γ. In terms of x and y, this relation takes the form

$$(xy^{-1})[x^3(xy^{-1})^{-12}]^n = 1.$$

In order to obtain the cardinality of $\widehat{R}(M_n)$, we now count the classes of representations $[\rho] \in \widehat{R}(K_{3,4})$ which satisfy this additional relation. (Note that if an element ρ of $R(K_{3,4})$ takes a relator to 1 in S^3, then so does any conjugate of ρ.)

If $[\rho] \in \widehat{R}(K_{3,4})$, then $\rho(x)^3 = \rho(y)^4 = \pm 1$. We consider first the case $\rho(x)^3 = \rho(y)^4 = +1$. In this case we must have $(\rho(x), \rho(y)) \in \Sigma_{-1/2} \times \Sigma_0$. (This is because these sets contain the only nontrivial 3rd and 4th roots of 1 in S^3.) To satisfy the additional relation, we insist in addition that

$$\rho(xy^{-1})[\rho(x)^3\rho(xy^{-1})^{-12}]^n = 1,$$

i.e., that $\rho(xy^{-1})^{1-12n} = 1$. Thus $\rho(xy^{-1})$ must be a $(12n-1)$th root of 1. Assume, by conjugation, that $\rho(y) = -\mathbf{i}$, so that $\rho(y^{-1}) = \mathbf{i}$. As $\rho(x)$ takes values in the 2-sphere $\Sigma_{-1/2}$, we see that $\rho(xy^{-1}) = \rho(x)\mathbf{i}$ takes values in the round but nonlatitudinal 2-sphere $(\Sigma_{-1/2})\mathbf{i}$.

Let $Q_{12n-1} = \{\sigma \in S^3 : \sigma^{12n-1} = 1\}$. Then we may write $Q_{12n-1} = \bigcup\{\Sigma_t : t = \cos(2\pi k/(12n-1)),\ k = 0, 1, \ldots, 6n-1\}$. Note that $(\Sigma_{(-1/2)}\mathbf{i}) \cap Q_{12n-1}$ is a disjoint union of circles, at most one of these circles lying in any latitudinal 2-sphere. The conjugacy classes of representations ρ in $\widehat{R}(M_n)$ are in one-to-one correspondence to the conjugacy classes (under simultaneous conjugation by an element of S^1) of pairs

$$(\rho(x), \rho(y)) \in \Sigma_{-1/2} \times \{-\mathbf{i}\}$$

satisfying $(\rho(x)\rho(y)^{-1})^{12n-1} = 1$. The number of conjugacy classes of these pairs is equal to the number of circles in $Q_{12n-1} \cap (\Sigma_{-1/2})\mathbf{i}$. The number of these circles is, in turn, equal to the number of $(12n-1)$th roots of 1 of the form $e^{i\theta}$, where $\pi/6 < \theta < 5\pi/6$. Thus we need simply to calculate the number of integers k satisfying the condition

$$\pi/6 < 2\pi k/(12n-1) < 5\pi/6.$$

An easy computation shows that there are precisely $4n$ such integers. Thus we have $4n$ classes of representations of $\pi_1(M_n)$ satisfying $\rho(x)^3 = \rho(y)^4 = 1$.

We now consider the case $\rho(x)^3 = \rho(y)^4 = -1$. In this case we must have $\rho(x) \in \Sigma_{1/2}$ and $\rho(y) \in \Sigma_{(1/\sqrt{2})} \cup \Sigma_{(-1/\sqrt{2})}$. After performing computations analogous to those in the first case, we find $3n$ classes of representations coming from

$$(\rho(x), \rho(y)) \in \Sigma_{1/2} \times \Sigma_{(1/\sqrt{2})},$$

and another $3n$ classes coming from

$$(\rho(x), \rho(y)) \in \Sigma_{1/2} \times \Sigma_{(-1/\sqrt{2})}.$$

Thus we obtain a total of $4n + 3n + 3n = 10n$ classes of representations of $\pi_1(M_n)$ in S^3. □

Note. The three sets of representations considered in this proof come from the three separate arcs of representations that make up $\widehat{R}(K_{3,4})$.

I.C. *m*-twist knots. The *m-twist knot* K_m is the knot whose projection is shown in Figure 3.

In this section we calculate the topological type of the representation space of the m-twist knot $(m = 1, 2, 3, \ldots)$.

Theorem 3. *$\widehat{R}(K_m)$ is the disjoint union of $[m/2]$ circles and, if m is odd, one open arc.*

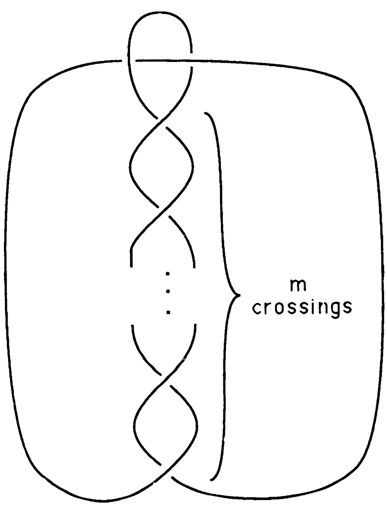


FIGURE 3. The m-twist knot K_m

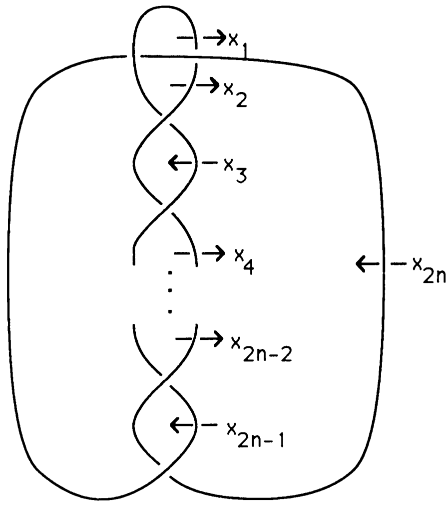


FIGURE 4. Wirtinger generators for $\pi_1(S^3 - K_m)$

Proof. We prove the theorem for the case in which m is even. The computations are completely analogous when m is odd. Define n by $m = 2n - 2$.

Using the pictured Wirtinger generators, we write:

$$
\begin{aligned}
\Gamma = \pi_1(K_m) = \{x_1, x_2, \dots, x_{2n} | & x_2^{-1} x_1 x_2 = x_3 \\
| & x_3 x_2 x_3^{-1} = x_4 \\
| & x_4^{-1} x_3 x_4 = x_5 \\
| & \quad \cdot \quad \cdot \\
| & \quad \cdot \quad \cdot \\
| & \quad \cdot \quad \cdot \\
| & x_{2n-1} x_{2n-2} x_{2n-1}^{-1} = x_{2n} \\
| & x_{2n}^{-1} x_1 x_{2n} = x_2\}.
\end{aligned}
$$

Notational convention. Suppose $\rho\colon \Gamma \to S^3$ is a nonabelian representation. Since the $\rho(x_i)$ are all conjugate to each other, they all lie in a common latitudinal Σ_t. From now on we will use x_i both for the generators of Γ, and for their images in Σ_t under ρ. For $1 \le i \le 2n - 1$, connect x_i to x_{i+1} in Σ_t by the shorter geodesic connecting them. (We will soon see that x_i and x_{i+1} are not antipodal, hence this procedure is not ambiguous.) This gives rise to a geodesic immersion of the following polygon, P_m, in Σ_t:

This immersion satisfies the following metric constraints:

(A) $d(x_i, x_{i+1}) = d(x_{i+1}, x_{i+2})$.
(B) $\angle x_i x_{i+1} x_{i+2} = -\angle x_{i+1} x_{i+2} x_{i+3}$.
(C) $d(x_2, x_{2n}) = d(x_1, x_{2n})$.

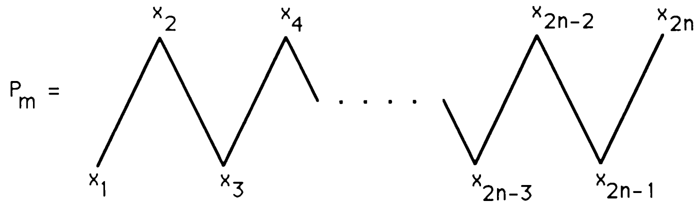


FIGURE 5. The polygon P_m

Because ρ is nonabelian, the angles in condition (B) are well defined and nonzero.

Define $\operatorname{imm}_*(P_m, \Sigma_0)$ to be the set of those geodesic immersions of P_m in Σ_0 which satisfy (A), (B), and (C).

Proposition 4. *There is a one-to-one correspondence between* $\operatorname{imm}_*(P_m, \Sigma_0)$ *and* $R(\Gamma) - S(\Gamma)$ (*i.e., nonabelian representations*).

Proof. Given a nonabelian representation ρ, the corresponding immersion is obtained by first connecting the points x_i and x_{i+1} of Σ_t by the shortest geodesic of Σ_t between them. There is no ambiguity; x_i and x_{i+1} cannot be antipodal. If they were, it would be impossible to satisfy the additional requirement

$$d(x_1, x_{2n}) = d(x_2, x_{2n}),$$

which is implied by the last relation. Now project Σ_t onto Σ_0 along the geodesics connecting $+1$ to -1 in S^3. This composition is the desired immersion. The fact that it satisfies the metric constraints (A), (B), and (C) is a consequence of the geometry of conjugation in S^3 discussed in §I.A.

Suppose, conversely, that we are given such an immersion $P_m \to \Sigma_0$. We then use projection along geodesics to map Σ_0 onto Σ_t, where

$$t = \cos(\tfrac{1}{2}\angle x_3 x_2 x_1).$$

By the geometry discussed in §I.A, it follows that we have a representation $\rho\colon \Gamma \to S^3$ as soon as we verify the following lemma.

Lemma 5. *If* P_{2n-2} *is immersed satisfying conditions* (A), (B), *and* (C) *in* Σ_0, *then it also satisfies*

$$\angle x_2 x_{2n} x_1 = \angle x_{2n-2} x_{2n-1} x_{2n}.$$

Proof of Lemma 5. In this proof, "$X = Y$" means that X is related to Y by an orientation-preserving isometry of Σ_0, where X and Y are any geometric figures. "$X = -Y$" means they are related by an orientation-reversing isometry.

By symmetry, it is clear that

$$\Delta x_{2n-1} x_{2n} x_2 = -\Delta x_{2n-2} x_{2n-1} x_1.$$

Hence,

$$\angle x_{2n-1} x_{2n} x_2 = -\angle x_{2n-2} x_{2n-1} x_1.$$

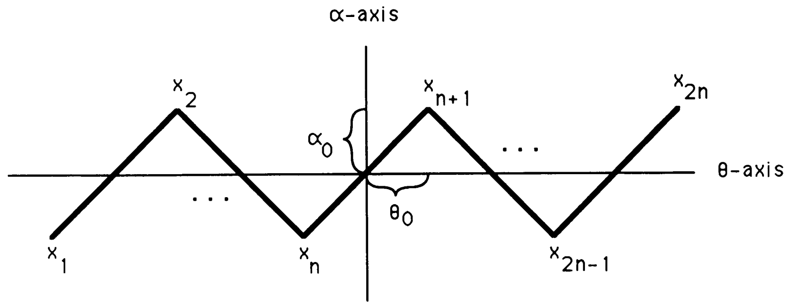


FIGURE 6. The image of P_m in a spherical coordinate system

By isosceles triangles,

$$\angle x_1 x_{2n} x_{2n-1} = -\angle x_1 x_{2n-1} x_{2n}.$$

Adding these equations gives

$$\angle x_1 x_{2n} x_2 = \angle x_{2n} x_{2n-1} x_{2n-2},$$

which completes the proof of the lemma and of the proposition. □

Continuing with the proof of Theorem 3, we will now give a canonical way to choose a representative of each oriented congruence class in $\mathrm{imm}_*(P_m, \Sigma_0)$. By elementary geometry (using symmetry), the midpoints of $x_1x_2, x_2x_3, \dots, x_{2n-1}x_{2n}$ all lie on and, in fact, determine a great circle in Σ_0. Consider the directed geodesic path σ that traces from midpoint(x_1x_2) to midpoint(x_2x_3) to $\cdots$ to midpoint$(x_{2n-1}x_{2n})$ (possibly traversing the great circle several times). Rotate the immersion so that σ lies on the **ij**-equator, midpoint(x_nx_{n+1}) lies at $\mathbf{i}$, and σ runs in the direction $\{\mathbf{i} \to \mathbf{j} \to -\mathbf{i}\}$. This prescription uniquely determines a representative of each oriented congruence class, and we shall say the resulting immersion is in "standard position".

Consider the coordinate system:

$$\mathbb{R} \times (-\pi/2, \pi/2) \to \Sigma_0$$

given by

$$(\theta, \alpha) \to \cos\alpha((\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}) + (\sin\alpha)\mathbf{k}.$$

Assuming our immersion is in standard position, we can pull it back to the $\theta\alpha$-plane to obtain the graph pictured in Figure 6.

We write **i-j-k** coordinates for some of these points in terms of the spherical coordinates α_0 and θ_0:

$$\begin{aligned} x_1 &= \cos(-\alpha_0)[\cos((1-2n)\theta_0)\mathbf{i} + \sin((1-2n)\theta_0)\mathbf{j}] + \sin(-\alpha_0)\mathbf{k}, \\ x_2 &= \cos(\alpha_0)[\cos((3-2n)\theta_0)\mathbf{i} + \sin((3-2n)\theta_0)\mathbf{j}] + \sin(\alpha_0)\mathbf{k}, \\ x_{2n} &= \cos(\alpha_0)[\cos((2n-1)\theta_0)\mathbf{i} + \sin((2n-1)\theta_0)\mathbf{j}] + \sin(\alpha_0)\mathbf{k}. \end{aligned}$$

Let $\langle\ ,\ \rangle$ denote the usual inner product on $\mathbb{R}^3 = \mathbb{R}\mathbf{i} + \mathbb{R}\mathbf{j} + \mathbb{R}\mathbf{k}$. Then for points x, y on Σ_0, we have $\langle x, y\rangle = \cos(\mathrm{dist}(x, y))$. Since $0 \le \mathrm{dist}(x, y) \le \pi$, the inner product $\langle x, y\rangle$ gives an unambiguous measure of distance. The

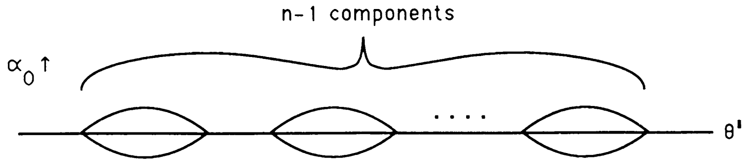


FIGURE 7. $\widehat{R}(K_m)$

immersion of P_m pulled back in Figure 6 automatically satisfies conditions A and B. We may assume that $-\pi/2 < \alpha_0 < \pi/2$ and that $0 \leq \theta_0 \leq \pi/2$ (by the use of shortest geodesics in constructing the immersion and the fact that x_n and x_{n+1} are not antipodal). We now calculate the conditions imposed on (θ_0, α_0) by condition C. Using the above formulae we compute:

$$\langle x_1, x_{2n}\rangle = \cos^2\alpha_0\cos((4n-2)\theta_0) - \sin^2\alpha_0,$$
$$\langle x_2, x_{2n}\rangle = \cos^2\alpha_0\cos((4n-4)\theta_0) + \sin^2\alpha_0.$$

Setting these equal, we obtain

$$(*) \qquad 2\tan^2\alpha_0 = \cos((2n-1)\theta') - \cos((2n-2)\theta'),$$

where $\theta' = 2\theta_0$ (hence we allow $0 \leq \theta' \leq \pi$).

Given a value of θ' for which the right-hand side (RHS) of $(*)$ is greater than zero, we obtain two different allowable values of α_0, and hence two allowable immersions. When $\mathrm{RHS}(*) = 0$, we obtain $\alpha_0 = 0$, hence a single allowable immersion. If $\mathrm{RHS}(*) < 0$, there is no solution. When we examine the right-hand side we find that $\mathrm{RHS}(*) \geq 0$ for values of θ' in $(n-1)$ closed subarcs of $[0, \pi]$. $\mathrm{RHS}(*) = 0$ precisely at the endpoints of these subarcs. Thus for each closed subarc we have two arcs of immersions, and they share endpoints, as shown in Figure 7.

Since each of these immersions corresponds to a distinct conjugacy class of representations, we find that, topologically,

$$\widehat{R}(K_m) = n-1 \text{ disjoint circles},$$

as was to be proved. □

Away from the points where $\alpha_0 = 0$, we can smoothly parametrize $\widehat{R}(K_m)$ using θ' as a parameter, because the derivative with respect to α_0 of $\mathrm{LHS}(*)$ is nonzero. When $\alpha_0 = 0$, the derivative with respect to θ' of $\mathrm{RHS}(*)$ is nonzero and, hence, near those points we may smoothly parametrize $\widehat{R}(K_m)$ using α_0 as a parameter. It follows that under this embedding in $\alpha_0\theta'$-space, $\widehat{R}(K_m)$ is a smooth 1-manifold.

I.D. The Whitehead link. The Whitehead link (WL) is the two-component link in S^3 with the projection shown in Figure 8.

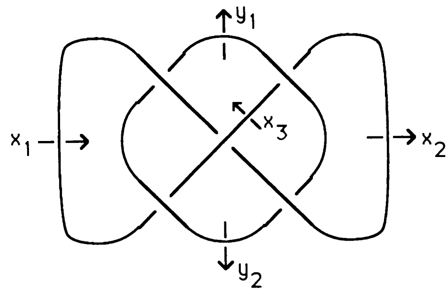


FIGURE 8. The Whitehead link WL

Using the pictured Wirtinger generators, its fundamental group is presented as follows:

$$\begin{aligned}\pi_1(S^3 - \mathrm{WL}) = \{x_1, x_2, x_3, y_1, y_2 | x_3x_1x_3^{-1} &= x_2, \\ | y_1x_2y_1^{-1} &= x_3, \\ | y_2^{-1}x_3y_2 &= x_1, \\ | x_2y_2x_2^{-1} &= y_1, \\ | x_1y_2x_1^{-1} &= y_1\}.\end{aligned}$$

Theorem 6. $\widehat{R}(\mathrm{WL})$ *is a punctured* $S^1 \times S^1$.

The proof of Theorem 6 will occupy the rest of this section. Let ρ: $\pi_1(S^3 - \mathrm{WL}) \to S^3$ be a representation.

Lemma 7. (a) $[\rho(x_1), \rho(y_1)] = 1 \Leftrightarrow \rho$ *is abelian.*
(b) $\rho(y_1) = \rho(y_2) \Leftrightarrow \rho$ *is abelian.*

Proof of Lemma. (a) ($\Leftarrow$) is obvious, so assume $[\rho(x_1), \rho(y_1)] = 1$. Then, by the relations, $\rho(y_1) = \rho(y_2)$ and $\rho(x_1) = \rho(x_2) = \rho(x_3)$, so ρ is abelian.
(b) Clearly, $[\rho(x_1), \rho(y_1)] = 1 \Leftrightarrow \rho(y_1) = \rho(y_2)$. Then, by (a), the lemma is proved. □

We use the fact that the conjugacy classes in S^3 are precisely the latitudinal 2-spheres Σ_t of constant real part t. Since y_1 and y_2 are conjugate, $\rho(y_1)$ and $\rho(y_2)$ lie in a single latitudinal 2-sphere. $\rho(x_1)$, $\rho(x_2)$, and $\rho(x_3)$ also lie in a single latitudinal 2-sphere.

We now examine the set of nonabelian representations, ρ. Up to conjugacy, $\rho(y_1)$ and $\rho(y_2)$ are determined by two real parameters: their real part t (the same for both), and their angular separation within the latitudinal 2-sphere Σ_t. We use one-half this separation as our second parameter α.

1st parameter: $-1 < t < 1$.
2nd parameter: $0 < \alpha \le \pi/2$.

Note. We exclude $t = \pm 1$ and $\alpha = 0$, because these would result in abelian representations by Lemma 7.

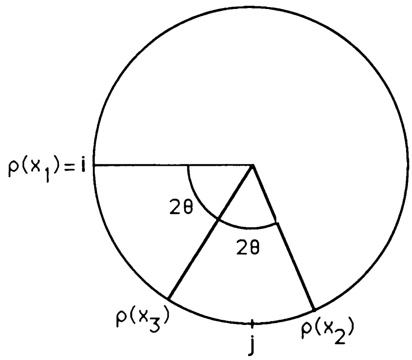


FIGURE 9. The **ij**-equator of Σ_0

Case 1. $\alpha = \pi/2$. In this case $\rho(y_1)$ and $\rho(y_2)$ are antipodal in their latitudinal 2-sphere. Since $x_1 y_2 x_1^{-1} = y_1$, we see that conjugation by $\rho(x_1)$ must correspond to rotation by 180°. It follows that $\rho(x_1)$ (and, hence, all $\rho(x_i)$) must lie in the equatorial 2-sphere Σ_0. We may assume (by conjugation) that

$$(\rho(y_1), \rho(y_2)) = (\cos\theta - \sin\theta\mathbf{k}, \cos\theta + \sin\theta\mathbf{k}),$$

where $t = \cos\theta$ and $0 < \theta < \pi$. Since $x_1 y_2 x_1^{-1} = y_1$, $\rho(x_1)$ must lie in the **ij**-equator of Σ_0. Thus we may assume (by conjugation) that $\rho(x_1) = \mathbf{i}$. Now, the relations $y_2^{-1} x_3 y_2 = x_1$ and $y_1 x_2 y_1^{-1} = x_3$ determine the positions of the $\rho(x_i)$, as pictured in Figure 9.

Thus for each of these antipodal positions of $(\rho(y_1), \rho(y_2))$ we obtain precisely one class of representations:

$$\rho(y_1) = \cos\theta - \sin\theta\mathbf{k}, \qquad \rho(y_2) = \cos\theta + \sin\theta\mathbf{k},$$
$$\rho(x_1) = \mathbf{i}, \qquad \rho(x_2) = \cos 4\theta\mathbf{i} + \sin 4\theta\mathbf{j},$$
$$\rho(x_3) = \cos 2\theta\mathbf{i} + \sin 2\theta\mathbf{j}.$$

It is easy to verify that these images satisfy the relations.

Case 2. $0 < \alpha < \pi/2$. By conjugation, assume $\rho(y_1)$ and $\rho(y_2)$ are in the following standard position:

$$\rho(y_1) = t + (1 - t^2)^{1/2}(\cos\alpha\mathbf{i} - \sin\alpha\mathbf{k}),$$
$$\rho(y_2) = t + (1 - t^2)^{1/2}(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{k}).$$

The following lemma gives information about the positions of the $\rho(x_i)$ in Case 2.

Lemma 8. *Let ρ be a nonabelian representation such that $(\rho(y_1), \rho(y_2))$ are in the above standard position, and $0 < \alpha < \pi/2$. Then there exist θ, β, and*

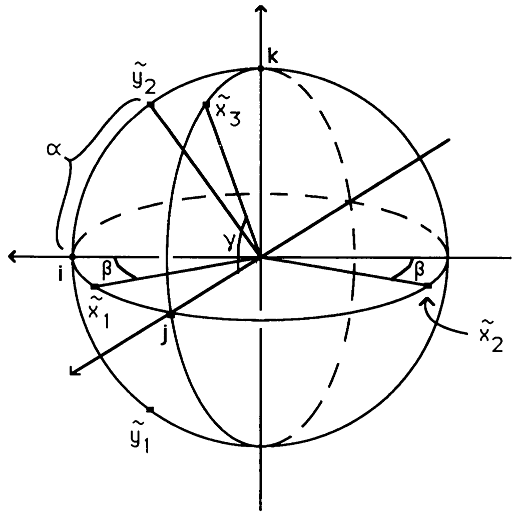


FIGURE 10. The points $\rho(x_i)$ and $\rho(x_j)$, as given in Lemma 8, projected onto Σ_0

γ *such that*

$$\rho(x_1) = \cos\theta + \sin\theta(\cos\beta\mathbf{i} + \sin\beta\mathbf{j}),$$
$$\rho(x_2) = \cos\theta + \sin\theta(-\cos\beta\mathbf{i} + \sin\beta\mathbf{j}),$$
$$\rho(x_3) = \cos\theta + \sin\theta(\cos\gamma\mathbf{j} + \sin\gamma\mathbf{k}).$$

Before continuing with the proof of this lemma, please consider Figure 10, a diagram of the relative positions of these points. In making this diagram, we have projected the images of all the generators onto Σ_0 along the geodesics connecting $+1$ to -1 in S^3. This enables us to see the images of all the generators on a single 2-sphere. We will refer to the images of the generators under this projection by the symbols $\tilde{x}_i$ and $\tilde{y}_i$.

Proof of Lemma 8. The relations $x_1y_2x_1^{-1} = y_1$ and $x_2y_2x_2^{-1} = y_1$ imply that $\tilde{x}_1$ and $\tilde{x}_2$ are each equidistant from $\tilde{y}_1$ and $\tilde{y}_2$; hence $\tilde{x}_1$ and $\tilde{x}_2$ both lie on the **ij**-equator. We also know that the angle of rotation about $\tilde{x}_1$ taking $\tilde{y}_2$ to $\tilde{y}_1$ is equal to the angle of rotation about $\tilde{x}_2$ taking $\tilde{y}_2$ to $\tilde{y}_1$. (This is because $\rho(x_1)$ and $\rho(x_2)$ both lie in the same latitudinal 2-sphere.) Once we have fixed $\tilde{x}_1$, this leaves only two possibilities for $\tilde{x}_2$. Either

(i) $\tilde{x}_1 = \tilde{x}_2$, or

(ii) $\tilde{x}_2$ is obtained from $\tilde{x}_1$ by reflecting through the **jk**-plane (i.e., changing the sign of its **i**-coordinate, as asserted in the lemma). Note that if we reflected $\tilde{x}_1$ through the **ik**-plane, we would get an angle of rotation of opposite sign.

We need to show that (ii) holds. If $\beta = \pm\pi/2$, then (i) and (ii) coincide, so assume $\beta \neq \pm\pi/2$. Suppose that (i) holds, so $\rho(x_1) = \rho(x_2)$. Then, since $x_3x_1x_3^{-1} = x_2$, we conclude that x_3 commutes with x_1. It follows that $\tilde{x}_3 = \pm\tilde{x}_1$ $(\rho(x_3) = \rho(x_1)^{\pm 1})$. If $\rho(x_3) = \rho(x_1)$, then $\rho(x_1)$ commutes with $\rho(y_1)$,

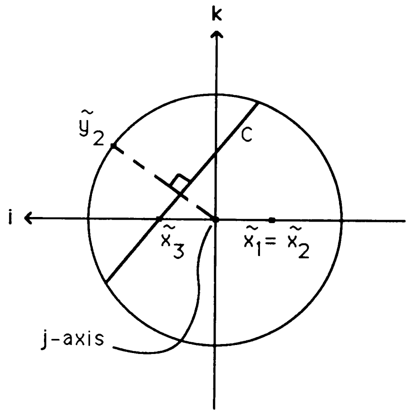


FIGURE 11. A projection of Σ_0 along the **j**-axis

so ρ is abelian by Lemma 7, giving a contradiction. If $\tilde{x}_3 = -\tilde{x}_1$, please consider Figure 11. This figure is a planar projection of Σ_0 along the **j**-axis. The equation $y_2 x_3 y_2^{-1} = x_1$ implies that $\tilde{x}_1$ is obtained from $\tilde{x}_3$ by a rotation about $\tilde{y}_2$. However, a rotation about $\tilde{y}_2$ must keep $\tilde{x}_3$ on the circle C (seen from the side as a line in the figure), which cannot intersect $\tilde{x}_1$! So we again have a contradiction. This eliminates possibility (i) above, and establishes that $\rho(x_1)$ and $\rho(x_2)$ are as claimed in Lemma 8.

All that remains is to show that $\tilde{x}_3$ lies on the **jk**-equator. If $\beta \neq \pm\pi/2$, then, since $x_3 x_1 x_3^{-1} = x_2$, $\tilde{x}_3$ is equidistant from $\tilde{x}_1$ and $\tilde{x}_2$; hence, $\tilde{x}_3$ is on the **jk**-equator. If $\beta = \pm\pi/2$, then $\tilde{x}_1 = \tilde{x}_2 = \pm\mathbf{j}$, so $\rho(x_3)$ commutes with $\rho(x_1)$. It follows that $\tilde{x}_3 = \pm\tilde{x}_1$, so $\tilde{x}_3$ is on the **jk**-equator. This completes the proof of Lemma 8. □

Using Lemma 8, we can associate to each nonabelian ρ such that $\rho(y_1) \neq \rho(y_2)^{-1}$ (i.e., in Case 2, above), a well-defined triple (α, β, γ), where $0 < \alpha < \pi/2$ and $\beta, \gamma \in \mathbb{R}/2\pi\mathbb{Z}$. Note that for $\rho(y_1) = \rho(y_2)^{-1}$ (i.e., in Case 1), we can also represent ρ by (α, β, γ) as in Lemma 8, with $\alpha = \pi/2$. However, instead of getting a well-defined triple, we get a choice of two possible triples, one with $\gamma = 0$ and one with $\gamma = \pi$. This is the only ambiguity. Thus we can identify $\widehat{R}(\mathrm{WL})$ with a subset of $\alpha\beta\gamma$-space, with identifications corresponding to the ambiguity in Case 1. We just need a simple criterion for deciding which triples (α, β, γ) correspond to representations. First, note that for (α, β, γ) to correspond with a representation, we must have either

(i) $\beta \in [-\pi/2, \pi/2]$ and $\gamma \in [0, \pi]$, or
(ii) $\beta \in [\pi/2, 3\pi/2]$ and $\gamma \in [-\pi, 0]$.

This can be seen from Figure 10. The fact that $\tilde{x}_1$ is related to $\tilde{x}_3$ by a rotation about $\tilde{y}_2$ implies that (i) or (ii) holds.

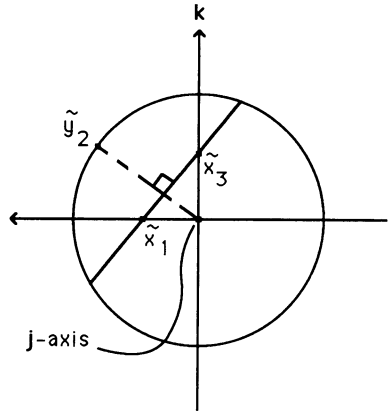


FIGURE 12. Another planar projection of Σ_0

Thus, our representations are to be sought in the union of two boxes: $(\alpha, \beta, \gamma) \in B_1 \cup B_2$, where

$$B_1 = (0, \pi/2] \times [-\pi/2, \pi/2] \times [0, \pi],$$
$$B_2 = (0, \pi/2] \times [\pi/2, 3\pi/2] \times [-\pi, 0].$$

Lemma 9. *If* $(\alpha, \beta, \gamma) \in B_1 \cup B_2$, *then* (α, β, γ) *corresponds to a representation* $\Leftrightarrow \cos\alpha\cos\beta = \sin\alpha\sin\gamma$.

Proof. (1) $d(\tilde{y}_2, \tilde{x}_1) = d(\tilde{y}_2, \tilde{x}_3) \Leftrightarrow$ (2) $\langle\tilde{y}_2, \tilde{x}_1\rangle = \langle\tilde{y}_2, \tilde{x}_3\rangle$, because these points lie on the unit 2-sphere. ($\langle\ ,\ \rangle$ is the usual inner product on $\mathbb{R}^3$.) When we write out (2) using the formulae in Lemma 8, we obtain precisely

$$\cos\alpha\cos\beta = \sin\alpha\sin\gamma.$$

It follows immediately from $y_2x_1y_2^{-1} = x_3$ that this condition is necessary. For sufficiency, assume we have (α, β, γ) satisfying

$$\cos\alpha\cos\beta = \sin\alpha\sin\gamma.$$

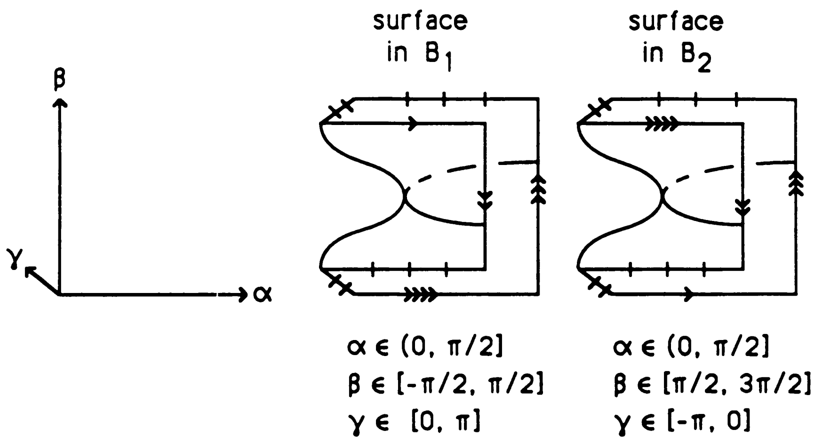
Then the angle of rotation about $\tilde{y}_2$ taking $\tilde{x}_3$ to $\tilde{x}_1$ determines the real part of the $\rho(y_i)$, and the angle of rotation about $\tilde{x}_3$ taking $\tilde{x}_1$ to $\tilde{x}_2$ determines the real part of the $\rho(x_i)$. It then follows from elementary geometric arguments that all the relations are satisfied, so we have a representation. This completes the proof of Lemma 9. □

We now set about analyzing the set of triples satisfying

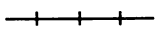
$$\cos\alpha\cos\beta = \sin\alpha\sin\gamma.$$

It is easy to verify that this equation defines smooth surfaces (in fact, discs) in B_1 and B_2. These surfaces in $\alpha\beta\gamma$-space are pictured in Figure 13. (Note that in order to allow a clear view of the boundaries of the surfaces, the perspective in these diagrams is somewhat distorted, so that the front face of each B_i is drawn smaller than the rear face. The γ-axis is intended to point out of the page, and the arrows on the ends of the coordinate axes point in the direction of higher coordinate values.)

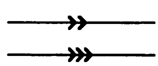
When we take the union of these two surfaces, while making the indicated identifications of their boundaries, we obtain $\widehat{R}(\mathrm{WL})$. It is left to the reader to verify that the resulting complex is a punctured torus. This completes the proof of the theorem. □



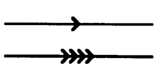
The meaning of the markings on the subarcs of the boundaries of the above discs is as follows:



These points correspond to abelian representations, hence are excluded.



These identifications are a consequence of the ambiguity in assigning a triple to a representation in Case 1 above.



These identifications are a consequence of equality $(\mathrm{mod}\, 2\pi)$ for β and for γ.

FIGURE 13. Two surfaces whose union (with identifications) is $\widehat{R}(\mathrm{WL})$

I.E. Binary dihedral representations. Define the binary dihedral group $N \subset S^3$ by $N = S^1_A \cup S^1_B$, where

$$S^1_A = \{a + b\mathbf{i} : a^2 + b^2 = 1\}$$

and

$$S^1_B = \{c\mathbf{j} + d\mathbf{k} : c^2 + d^2 = 1\}.$$

Topologically, N is the disjoint union of two circles. S^1_A is a subgroup of N of index 2. Let $K \subset S^3$ be a knot. Let $\Gamma = \pi_1(S^3 - K)$ have Wirtinger presentation

$$\Gamma = \{x_1, \dots, x_n : (x_{j_i}^{\varepsilon_i})x_i(x_{j_i}^{-\varepsilon_i}) = x_{i+1}, \ 1 \leq i \leq n-1\},$$

where $\varepsilon_i = \pm 1$ for all i. The following theorem is the main result of this section.

Theorem 10. *The number of conjugacy classes of nonabelian homomorphisms $\Gamma \to N$ is equal to*

$$(|\Delta_K(-1)| - 1)/2,$$

where $\Delta_K(t)$ is the Alexander polynomial of K.

Note that $\Delta_K(-1)$ is always an odd integer (see, for example, Rolfsen [Ro, p. 213]).

Corollary 11. *If $|\Delta_K(-1)| \neq 1$, then there exist nonabelian representations of $\pi_1(S_3 - K)$ into $N \subset S^3$.*

As an example the torus knots of type (p, q), where p and q are both odd, are knots without nonabelian representations into N. The groups of these torus knots do, however, have nonabelian representations into S^3.

Proof of Theorem 10. The idea of this proof is closely related to the study of metacyclic representations in Fox [F, pp. 160–163].

Suppose $\rho \colon \Gamma \to N$ has nonabelian image. Since the x_i are all conjugate, and since S^1_A is normal in N, we know that either all the $\rho(x_i)$ are contained in S^1_A or all are contained in S^1_B. If $\{\rho(x_i)\}$ is contained in S^1_A then ρ is abelian, so assume $\{\rho(x_i)\}$ is contained in S^1_B. Each element of S^1_B can be expressed as

$$e^{\mathbf{i}\theta}\mathbf{j} = \cos\theta\mathbf{j} + \sin\theta\mathbf{k}$$

for some $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Any two elements of S^1_B are conjugate in N because

$$e^{\mathbf{i}\alpha}(e^{\mathbf{i}\theta}\mathbf{j})e^{-\mathbf{i}\alpha} = e^{\mathbf{i}(\theta+2\alpha)}\mathbf{j}.$$

Hence, by conjugation, we can assume that $\rho(x_n) = \mathbf{j}$. For each i, $1 \leq i \leq n$, define $\theta_i \in \mathbf{R}/2\pi\mathbb{Z}$ by $\rho(x_i) = e^{\mathbf{i}\theta_i}\mathbf{j}$.

To see what conditions are imposed on the θ_i by the relations in our presentation for Γ, we substitute this expression for $\rho(x_i)$ into those relations. The result is the following system of linear equations over $\mathbb{R}/2\pi\mathbb{Z}$ for the θ_i:

$$\theta_i + \theta_{i+1} - 2\theta_{j_i} = 0,$$

where $1 \le i \le n-1$ and $\theta_n = 0$. It follows that the set of n-tuples

$$(\theta_1, \theta_2, \dots, \theta_{n-1}, 0) \in (\mathbb{R}/2\pi\mathbb{Z})^n$$

satisfying this system of equations is in one-to-one correspondence with the set J of representations of Γ into N taking x_n to $\mathbf{j}$. In the ith row of the $(n-1)\times n$ matrix corresponding to this system, we have the following nonzero entries:

$$\begin{array}{rccc} \text{column}: & i & i+1 & j_i \\ \text{entry}: & 1 & 1 & -2 \end{array}$$

We wish to count the set of solutions to this system over $\mathbb{R}/2\pi\mathbb{Z}$ having $\theta_n = 0$; thus we may drop the last column of the matrix. The ith row of the resulting matrix, which we call A, is the same, up to sign, as the ith row of $A_K(-1)$, where $A_K(t)$ is obtained from the Alexander matrix corresponding to our original group presentation by removing its last column. Since

$$\det A = \pm\Delta_K(-1)$$

is always an odd integer, we know A is nonsingular.

Think of A as a linear transformation $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. The number of solutions $\bmod 2\pi\mathbb{Z}$ is equal to

$$\begin{aligned} &\operatorname{card}\{A^{-1}(2\pi\mathbb{Z}^{n-1})/(2\pi\mathbb{Z}^{n-1})\} \\ &\quad = \operatorname{card}\{(2\pi\mathbb{Z}^{n-1})/A(2\pi\mathbb{Z}^{n-1})\} \\ &\quad = \operatorname{card}(\mathbb{Z}^{n-1}/A(\mathbb{Z}^{n-1})) = |\det(A)| = |\Delta_K(-1)|. \end{aligned}$$

There is precisely one abelian representation taking $x_n \to \mathbf{j}$, namely, the one taking all $x_i \to \mathbf{j}$. Each nonabelian representation in J has precisely one distinct conjugate in J: the one obtained from conjugating by $\mathbf{j}$. It follows that the number of conjugacy classes of nonabelian representations is $(|\Delta_K(-1)| - 1)/2$. This completes the proof of Theorem 10. □

I.F. Incompressible surfaces and $\dim(\widehat{R}(K))$. In this section we explore the relationship between incompressible surfaces in knot complements, and higher dimensional components in $\widehat{R}(K)$. We begin by using composite and doubled knots to show how incompressible annuli and tori can lead to components of $\widehat{R}(K)$ whose dimension is greater than one. A proposition is then proved which states that a component of $\widehat{R}(K)$ whose dimension is greater than one always leads to closed, nonboundary parallel, incompressible surfaces in the complement of K.

Composite knots. Let $K \subset S^3$ be a knot, and let $\mu \in \pi_1(S^3 - K)$ be a group element corresponding to a meridian of K. Define $\varphi_K \colon \widehat{R}(K) \to [-1, 1]$ by $\varphi_K([\rho]) = \mathrm{Re}(\rho(\mu))$. Define $\widehat{R}_t(K) = \varphi_K^{-1}(t) = \{[\rho] : \rho(\mu) \in \Sigma_t\}$.

Let K_1 and K_2 be oriented knots in S^3, and let $K = K_1 \# K_2$ denote their connected sum. Let $\mu_i \in \pi_i(S^3 - K_i)$, $i = 1, 2$, be meridians of these knots. Van Kampen's theorem allows us to express $\pi_1(S^3 - K)$ as a free product with amalgamation

$$\pi_1(S^3 - K) = \pi_1(S^3 - K_1) *_f \pi_1(S^3 - K_2),$$

where the amalgamating homomorphism $f \colon \langle\mu_1\rangle \to \langle\mu_2\rangle$ is given by $f(\mu_1) = \mu_2$. Let $\mu = \mu_1 = \mu_2 \in \pi_1(S^3 - K)$ be a meridian of K. The following proposition expresses $\widehat{R}_t(K)$ in terms of $\widehat{R}_t(K_1)$ and $\widehat{R}_t(K_2)$.

Proposition 12. $\widehat{R}_t(K) = X_1 \cup X_2 \cup X_{12}$ *(a disjoint union), where*

$$X_1 \cong \widehat{R}_t(K_1), \qquad X_2 \cong \widehat{R}_t(K_2),$$

and there is a surjective map

$$\psi \colon X_{12} \to \widehat{R}_t(K_1) \times \widehat{R}_t(K_2)$$

with the property that $\psi^{-1}([\rho_1], [\rho_2])$ *is a circle for all* $[\rho_1] \in \widehat{R}_t(K_1)$ *and* $[\rho_2] \in \widehat{R}_t(K_2)$.

Proof. The intuitive idea behind this proof is that the circles which make up X_{12} are obtained by pivoting representations of $\pi_1(S^3 - K_2)$ about representations of $\pi_1(S^3 - K_1)$ using conjugation in S^3.

Suppose $\rho \in R(K)$ is nonabelian. Define $\rho_i \in R(K_i)$ by $\rho_i = \rho|\pi_1(S^3 - K_i)$ for $i = 1, 2$. We then write $\rho = \rho_1 * \rho_2$. Conversely, given $\rho_1 \in R(K_1)$ and $\rho_2 \in R(K_2)$, we can form $\rho = \rho_1 * \rho_2$ if and only if $\rho_1(\mu_1) = \rho_2(\mu_2)$. Now define

$$X_1 = \{[\rho_1 * \rho_2] \in \widehat{R}_t(K) : \rho_2 \text{ is abelian}\},$$
$$X_2 = \{[\rho_1 * \rho_2] \in \widehat{R}_t(K) : \rho_1 \text{ is abelian}\},$$

and

$$X_{12} = \{[\rho_1 * \rho_2] \in \widehat{R}_t(K) : \rho_1 \text{ and } \rho_2 \text{ are nonabelian}\}.$$

Clearly, $X_1 \cup X_2 \cup X_{12}$ is a disjoint union.

Note. If ρ_1 and ρ_2 are both abelian, then so is $\rho_1 * \rho_2$. However, we are assuming $\rho_1 * \rho_2$ is nonabelian, so it follows that either ρ_1 or ρ_2 is nonabelian.

Given ρ_1 with $[\rho_1] \in \widehat{R}_t(K_1)$, there is a unique abelian representation $\rho_2 \colon \pi_1(S^3 - K_2) \to S^3$ satisfying $\rho_1(\mu_1) = \rho_2(\mu_2)$. It follows that X_1 is homeomorphic to $\widehat{R}_t(K_1)$ and, by the same argument, that X_2 is homeomorphic to $\widehat{R}_t(K_2)$.

Define the map $\psi\colon X_{12} \to \widehat{R}_t(K_1) \times \widehat{R}_t(K_2)$ by

$$\psi([\rho_1 * \rho_2]) = ([\rho_1], [\rho_2]).$$

One easily checks that ψ is well defined. To see that ψ is surjective, observe that if $([\rho_1], [\rho_2]) \in \widehat{R}_t(K_1) \times \widehat{R}_t(K_2)$, then there exists $\sigma \in S^3$ such that $\sigma\rho_2(\mu_2)\sigma^{-1} = \rho_1(\mu_1)$. Hence $\psi([\rho_1 * (\sigma\rho_2\sigma^{-1})]) = ([\rho_1], [\rho_2])$.

To see that point inverses under ψ are circles, note that the set of all cosets $[\tau] \in S^3/\{\pm 1\}$ satisfying $\tau\rho_2(\mu_2)\tau^{-1} = \rho_1(\mu_1)$ is a coset of the circle subgroup containing $[\rho_2(\mu_2)] \in S^3/\{\pm 1\}$ and is, therefore, a circle. Since $S^3/\{\pm 1\}$ acts freely on the nonabelian representations, this implies that $\psi^{-1}([\rho_1], [\rho_2])$ is a circle. □

If $t \in \operatorname{im}(\varphi_{K_1}) \cap \operatorname{im}(\varphi_{K_2})$ then, by the previous proposition, $\dim(\widehat{R}_t(K)) \geq 1$. Suppose, in addition, that $[\rho_1] \in \varphi_{K_1}^{-1}(t)$ is a regular point of φ_{K_1}. By this, we mean that we can find a smooth arc A_1 in $\widehat{R}(K_1)$ containing $[\rho_1]$ such that the derivative at $[\rho_1]$ of $\varphi_{K_1}|A_1$ is nonzero. Suppose that $[\rho_2] \in \varphi_{K_2}^{-1}(t)$ is a regular point of φ_{K_2} in the same sense. Assume, by restricting to smaller arcs if necessary, that the derivative of $\varphi_{K_i}|A_i$ is nonzero for $i = 1, 2$. Define

$$D = \{([\rho_1], [\rho_2]) \in A_1 \times A_2 | \varphi_{K_1}[\rho_1] = \varphi_{K_2}[\rho_2]\},$$

a smooth arc in $A_1 \times A_2$. Then $\psi\colon \psi^{-1}(D) \to D$ is a submersion, and since $\psi^{-1}([\rho_1], [\rho_2])$ is a circle, it follows that $\psi^{-1}(D)$ is 2-dimensional. The preceding discussion proves the following proposition.

Proposition 13. *Let K_1 and K_2 be two knots in S^3. Suppose $[\rho_1] \in \widehat{R}(K_1)$ and $[\rho_2] \in \widehat{R}(K_2)$ are regular points of φ_{K_1} and φ_{K_2}, respectively, and that $\varphi_{K_1}[\rho_1] = \varphi_{K_2}[\rho_2]$. Then $[\rho_1 * \rho_2]$ is contained in a 2-dimensional component of $\widehat{R}(K_1 \# K_2)$.*

In fact, using Proposition 12, together with a knowledge of $\widehat{R}(K_1)$ and $\widehat{R}(K_2)$, we can form an accurate picture of $\widehat{R}(K_1 \# K_2)$. We illustrate this in the following two examples.

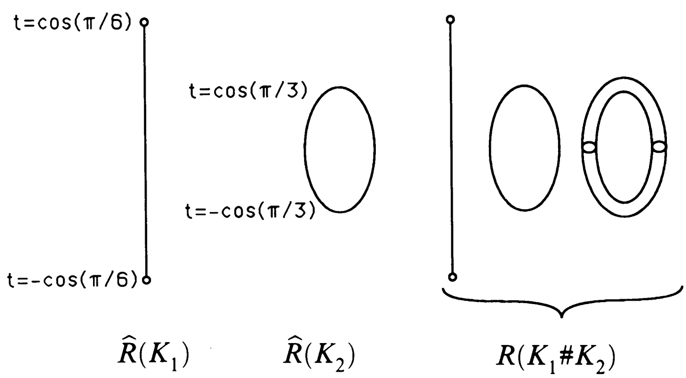


FIGURE 14. The representation space of the connected sum of the trefoil and figure 8 knots

Example 1. Let K_1 be the trefoil knot and K_2 the figure eight knot. (In the notation of §I.C, K_2 is the 2-twist knot.) In Figure 14, we have diagrammed $\widehat{R}(K_1)$ and $\widehat{R}(K_2)$.

In Figure 14, each representation space is pictured in such a way that the vertical coordinate is $t = \mathrm{Re}(\rho(\mu))$. From our pictures of $\widehat{R}(K_1)$ and $\widehat{R}(K_2)$, we deduce the pictures of $\widehat{R}(K_1 \# K_2)$ as follows. For each value of t, use Proposition 12 to construct $\widehat{R}_t(K_1 \# K_2)$ from $\widehat{R}_t(K_1)$ and $\widehat{R}_t(K_2)$. Then piece together these level sets to obtain $\widehat{R}(K_1 \# K_2)$. (Figure 14 is inaccurate inasmuch as each meridian curve of the torus component should be horizontal, i.e., should correspond to a fixed value of t.)

Example 2. $K = K_{5,2} \# K_{3,2}$ (where $K_{p,q}$ is a (p, q)-torus knot).

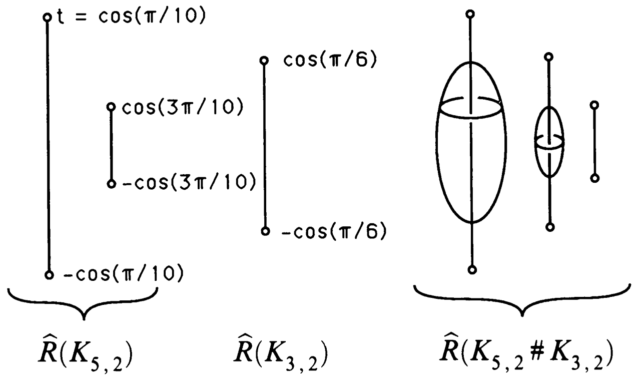


FIGURE 15. The representation space of the connected sum of two torus knots

In studying Figure 15, recall that the endpoints of the arcs of nonabelian representations in $\widehat{R}(K_{5,2})$ and $\widehat{R}(K_{3,2})$ correspond to abelian representations. For this reason, for example, as $t \to \cos(\pi/6)^-$, the family of circles corresponding to X_{12} approaches a single point of R_1. This results in the existence of singular points (i.e., nonmanifold points) in $\widehat{R}(K_{5,2} \# K_{3,2})$.

Doubled knots. Let L be the knot contained in the solid torus V pictured in Figure 16. Let $\overline{\mu}$ and $\overline{\lambda}$ be the oriented meridian and longitude curves in ∂V pictured in Figure 16.

Let $K \subset S^3$ be a knot, and $N(K)$ a closed tubular neighborhood of K in S^3. Let μ and λ in $\partial N(K)$ represent an oriented meridian and a preferred longitude of $K \subset S^3$. If n is an integer, there exists a homeomorphism

$$\varphi_n \colon \partial N(K) \to \partial V$$

satisfying $\varphi_n(\mu) = \overline{\mu}$ and $\varphi_n(\lambda) = \overline{\lambda} + n\overline{\mu}$, where, as usual, we are also letting μ, λ, etc., represent their corresponding fundamental group elements. These properties determine φ_n up to isotopy. $[S^3 - \mathrm{int}(N(K))] \cup_{\varphi_n} V$ is homeomorphic to S^3. Hence, we may think of L as a knot K_n in this 3-sphere; we will refer to K_n as the *n-twisted double* of K.

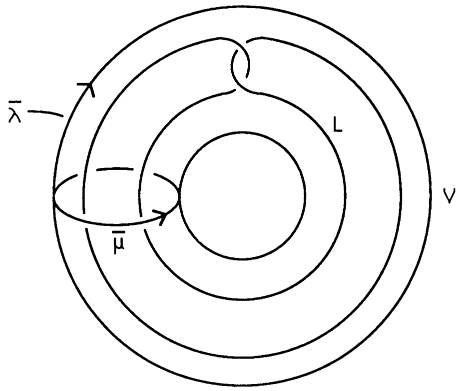


FIGURE 16. The knot L in the solid torus V

From the construction described above, it is clear that the complement of the n-twisted double of K can be expressed as

$$S^3 - K_n = [S^3 - \operatorname{int}(N(K))] \cup_{\varphi_n} [V - L].$$

By Van Kampen's theorem, we may express $\pi_1(S^3 - K_n)$ as a free product with amalgamation:

$$\pi_1(S^3 - K_n) = \pi_1(S^3 - \operatorname{int}(N(K))) *_{(\varphi_n)_*} \pi_1(V - L).$$

We now show that, generically, $\widehat{R}(K_n)$ has 2-dimensional components. Observe that $V - L \cong S^3 - \mathrm{WL}$, where WL is the Whitehead link, defined in §I.D. It follows that each representation $\rho \in R(K_n)$ can be expressed in the form $\rho = \rho_1 * \rho_2$, where $\rho_1 \in R(K)$, $\rho_2 \in R(\mathrm{WL})$, and they satisfy

$$\rho_1(\mu) = \rho_2(\varphi_n(\mu)) \quad \text{and} \quad \rho_1(\lambda) = \rho_2(\varphi_n(\lambda)).$$

Define $\widetilde{A} = \{(x, y) \in S^3 \times S^3 : [x, y] = 1\}$. S^3 acts by conjugation on $\widetilde{A}$. Let $A = \widetilde{A}/(\text{action by } S^3)$. Since

$$A \cong S^1 \times S^1/(x, y) \sim (\overline{x}, \overline{y}),$$

A is a 2-dimensional orbifold. We now define maps $q_1 \colon \widehat{R}(K) \to A$ by $q_1([\rho]) = [\rho(\mu), \rho(\lambda)]$, and $q_2 \colon \widehat{R}(\mathrm{WL}) \to A$, by $q_2([\rho]) = [\rho(\varphi_n(\mu)), \rho(\varphi_n(\lambda))]$. Let $[\rho_1] \in \widehat{R}(K)$ and $[\rho_2] \in \widehat{R}(\mathrm{WL})$. We can form $[\rho_1 * \rho_2] \in \widehat{R}(K_n)$ if and only if $q_1[\rho_1] = q_2[\rho_2]$.

We have used computations involving $\widehat{R}(\mathrm{WL})$, as computed in §I.D, to show that q_2 is a submersion on the complement of a set of measure zero in $\widehat{R}(\mathrm{WL})$. We will omit these computations from the present paper, and will present them, instead, in a forthcoming paper on doubled knots. (See [K].)

Now suppose, as for the examples computed in §§1.B and I.C, that $\widehat{R}(K)$ is a 1-manifold. Then, generically, one expects $q_1^{-1}(\operatorname{im} p_2)$ to contain 1-dimensional submanifolds of $\widehat{R}(K)$. Let $M \subset q_1^{-1}(\operatorname{im} q_2)$ be such a submanifold. By definition, for each $\rho_1 \in M$, there exists a $\rho_2 \in \widehat{R}(\mathrm{WL})$ such that $q_1[\rho_1] = q_2[\rho_2]$. By conjugating ρ_2 by an element of S^3, we may assume that $\rho_1(\mu) = \rho_2(\varphi_n(\mu))$ and $\rho_1(\lambda) = \rho_2(\varphi_n(\lambda))$. For each such pair $([\rho_1], [\rho_2])$, we can form a circle of classes in $\widehat{R}(K_n)$ of the form $[\rho_1 * (\sigma\rho_2\sigma^{-1})]$, where σ varies in the circle subgroup containing $\rho_1(\mu)$ and $\rho_1(\lambda)$. Since $\widehat{R}(K_n)$ contains such a circle for each point in the 1-manifold M, it follows that for this generic case $\widehat{R}(K_n)$ contains 2-dimensional components. Thus we have the following proposition.

Proposition 14. *Let $K \subset S^3$ be a knot. Suppose that $\dim \widehat{R}(K) \geq 1$. Then, subject to the genericity assumption in the preceding paragraph,* $\dim \widehat{R}(K_n) \geq 2$. □

We have carried out this computation for $K_n =$ the untwisted double of the trefoil, and in this case $\widehat{R}(K_n) =$ the union of four tori. For more information, see [K].

A proposition relating higher-dimensional components of $\widehat{R}(K)$ to incompressible surfaces. Thus far in this section we have seen that if K is a composite or a doubled knot, $\widehat{R}(K)$ generally contains components of dimension greater than one. Note that these two types of knots are examples of nontrivial satellite knots, which we define to be knots whose complements contain nonboundary parallel incompressible tori. We now prove a partial converse to Propositions 13 and 14.

Proposition 15. *Let $K \subset S^3$ be a knot and $N(K)$ a closed tubular neighborhood of K in S^3. If $\widehat{R}(K)$ contains a component of dimension greater than or equal to two, then $S^3 - \mathrm{int}(N(K))$ contains a nonempty system of closed, nonboundary parallel, incompressible surfaces.*

Proof. This proposition, the proof of which will occupy the rest of this section, is a direct consequence of results in Culler and Shalen's paper *Varieties of representations and splittings of* 3*-manifolds* [CS], and we shall follow their notation as much as possible. Define $X(S^3 - K)$ to be the set of characters of representations of $\pi_1(S^3 - K)$ in $\mathrm{SL}(2, \mathbb{C})$. Culler and Shalen show that $X(S^3 - K)$ can be given the structure of a complex affine algebraic set. Its ambient coordinates are given by $\{\chi(g_i)\}$, where $\chi \in X(S^3 - K)$ is a character and $\{g_i\}$ is a finite subset of $\pi_1(S^3 - K)$. Because $\mathrm{SU}(2)$ is a subgroup of $\mathrm{SL}(2, \mathbb{C})$, there is an obvious map

$$t\colon R(K) \to X(S^3 - K),$$

which associates to each representation the corresponding character. We claim that this map induces an injection, which we call $\hat{t}$:

$$\hat{t}\colon \widehat{R}(K) \to X(S^3 - K).$$

This claim will follow from the following two facts:

(1) Two irreducible representations in $\mathrm{SL}(2, \mathbb{C})$ are conjugate if and only if they correspond to the same character.

(2) If two irreducible (i.e., nonabelian) representations in $\mathrm{SU}(2)$ are conjugate by an element of $\mathrm{SL}(2, \mathbb{C})$, then they are conjugate by an element of $\mathrm{SU}(2)$.

Culler and Shalen prove (1) [CS, Proposition 1.5.2].

Proof of (2). Let $\rho_1, \rho_2 \colon \Gamma \to \mathrm{SU}(2)$ be two nonabelian representations and suppose that $A\rho_1 A^{-1} = \rho_2$, where $A \in \mathrm{SL}(2, \mathbb{C})$. Assume that A is not in $\mathrm{SU}(2)$. Think of the elements of $\mathrm{SL}(2, \mathbb{C})$ as acting on hyperbolic 3-space, H^3, via the double cover $\mathrm{SL}(2, \mathbb{C}) \to \mathrm{PSL}(2, \mathbb{C})$. Then $\mathrm{SU}(2)$ is just the subgroup of $\mathrm{SL}(2, \mathbb{C})$ fixing a particular point $p \in H^3$. Since $\rho_1(\Gamma)$ fixes p, $\rho_2(\Gamma) = A\rho_1(\Gamma)A^{-1}$ fixes $A(p)$. Since A is not in $\mathrm{SU}(2)$, $A(p) \neq p$. Since $\rho_2(\Gamma) \in \mathrm{SU}(2)$ fixes two distinct points, p and $A(p)$, it fixes an entire hyperbolic line. It follows that $\rho_2(\Gamma)$ is abelian, a contradiction that implies that our assumption that A is not in $\mathrm{SU}(2)$ was false. □

Having proven the claim, we identify $\widehat{R}(K)$ with a subset of $X(S^3 - K)$ via the map $\hat{t}$. Since traces of elements of $\mathrm{SU}(2)$ are real, $\widehat{R}(K)$ is made up of real points of $X(S^3 - K)$.

Lemma 16. *If $\widehat{R}(K)$ has a component of real dimension greater than or equal to two, then $X(S^3 - K)$ has a component of complex dimension greater than or equal to two.*

Proof. Let V be a component of $\widehat{R}(K)$ such that $\dim_{\mathbb{R}}(V) \geq 2$. Let $[\rho] \in V$ be a regular (i.e., nonsingular) point. Then there is a map $f \colon \mathbb{R}^2 \to V$, given by a power series, which is an embedding near 0 (i.e., its first partials are nonzero), and which satisfies $f(0) = [\rho]$. Since the power series for f formally satisfies the polynomials defining $X(S^3 - K)$, we may extend f to a function $\mathbb{C}^2 \to X(S^3 - K)$ given by the same power series. This extension, which we also call f, is also locally an embedding, since its first partials are nonzero. It follows that $[\rho]$ is contained in a component of $X(S^3 - K)$ of complex dimension greater than or equal to two. □

Hatcher, in [H], proves that only a finite number of isotopy classes of simple closed curves in $\partial N(K)$ can occur as boundary components of properly embedded incompressible surfaces in $S^3 - \mathrm{int}(N(K))$. We may conclude that there exists a simple closed curve γ in $\partial N(K)$ which does not occur as one of the boundary components of any incompressible surface. We use γ to denote the corresponding element of $\pi_1(S^3 - \mathrm{int}(N(K)))$, as well. Let $\widetilde{V} \subset X(S^3 - K)$ denote a component with $\dim_{\mathbb{C}}(\widetilde{V}) \geq 2$ (which we have just shown to exist). Define $I_\gamma \colon \widetilde{V} \to \mathbb{C}$ by $I_\gamma(\chi) = \chi(\gamma)$. Let $C \subset \widetilde{V}$ be a complex affine curve on which I_γ is constant. Let $\widetilde{C}$ be a desingularized projective curve with function field isomorphic to that of C (see [CS] for details). Let $\tilde{x}$ be an ideal point of $\widetilde{C}$. Since I_γ is constant on C, the corresponding function $\widetilde{I}_\gamma$ on $\widetilde{C}$ does not have a pole at $\tilde{x}$. It follows from Theorems 2.2.1 and 2.3.1 of [CS] that we can associate to $\tilde{x}$ a nonempty system $\Sigma = S_1 \cup \cdots \cup S_n$ of properly embedded, nonboundary parallel, incompressible surfaces in $S^3 - \operatorname{int}(N(K))$ such that $\gamma \cap \Sigma = \varnothing$. Since any boundary circles of S_i must be disjoint from γ, they must be parallel to and, hence, in the same isotopy class as γ. Since this contradicts the choice of γ not to be a boundary curve, we conclude that Σ is a system of closed incompressible surfaces, completing the proof of Proposition 15. □

II. Tangent space computations

II.A. The Zariski tangent space to $R(\Gamma)$. Let Γ be a finitely presented group with presentation

$$\Gamma = \{x_1, x_2, \ldots, x_n : w_1, w_2, \ldots, w_m\}.$$

In this section we set up a framework (due to Weil [W]) for computing the Zariski tangent space to $R(\Gamma)$ at a particular representation ρ.

Let $\mathbf{g}$ be the Lie algebra of S^3. If $q \in S^3$, recall that $C_q\colon S^3 \to S^3$ is given by $C_q(\sigma) = q\sigma q^{-1}$. Let $(dC_q)_1$ denote the differential of C_q at the point $1 \in S^3$. Define $\operatorname{Ad}\colon S^3 \to \operatorname{Aut}(\mathbf{g})$ by

$$\operatorname{Ad}(q)(z) = (dC_q)_1(z) \quad \text{for } z \in \mathbf{g}.$$

If $\rho \in R(\Gamma)$, define $\tilde{\rho}\colon \Gamma \to \operatorname{Aut}(\mathbf{g})$ by

$$\tilde{\rho}(x)(z) = \operatorname{Ad}(\rho(x))(z), \qquad x \in \Gamma,\ z \in \mathbf{g}.$$

$\tilde{\rho}$ is known as the adjoint representation to ρ.

Lubotzky and Magid [LM, p. 62] show that we may calculate the scheme-theoretic Zariski tangent space, $T_\rho R(\Gamma)$, to $R(\Gamma)$ at a representation ρ, by the formula

$$T_\rho R(\Gamma) = \bigcap \{\ker(dw_i)_\rho : i = 1, \ldots, m\}.$$

(In this formula, we are thinking of $R(\Gamma)$ as a subset of $(S^3)^n$, by identifying ρ with the point $(\rho(x_1), \ldots, \rho(x_n))$ in $(S^3)^n$. The words w_i are being considered as maps from $(S^3)^n$ to S^3, as discussed in §I.A.) We shall need an algorithm for computing this tangent space. The formula (Proposition 18) that we derive appears in Weil [W, p. 151], and is also derived by Lubotzky and Magid in [LM]. For convenience, a slightly different derivation is included here.

We begin by sharpening our notation. Define

$$q_i\colon (S^3)^n \to S^3$$

to be projection on the ith factor, for $i = 1, \ldots, n$. Define

$$q_i^{-1}\colon (S^3)^n \to S^3$$

to be q_i followed by inversion in the group S^3. Suppose we are given a word w in the n letters $\{x_1, \dots, x_n\}$,

$$w = x_{j_1}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k},$$

where $\varepsilon_i = \pm 1$ and $1 \le j_i \le n$, for $i = 1, \dots, k$. The corresponding map $w: (S^3)^n \to S^3$ is then given by the product

$$w = q_{j_1}^{\varepsilon_1} q_{j_2}^{\varepsilon_2} \cdots q_{j_k}^{\varepsilon_k}.$$

S^3 acts on its own tangent bundle from the left and from the right by translation. We indicate this action by the appropriate juxtaposition σv or $v\sigma$, where $v \in T_* S^3$ and $\sigma \in S^3$. Let

$$\sigma = (\sigma_1, \dots, \sigma_n) \in (S^3)^n.$$

We wish to compute the composition

$$\mathbf{g}^n \to T_\sigma((S^3)^n) \to T_{w(\sigma)}(S^3) \to \mathbf{g},$$

where the first and last maps are right translations, and the middle map is dw_σ. For the purpose of the following lemma, we will denote this composition by $d\tilde{w}_\sigma$; thereafter, by abuse of notation, we will always denote it simply dw_σ.

Lemma 17. *Let* $z = (z_1, \dots, z_n) \in \mathbf{g}^n$. *Then*

$$d\tilde{w}_\sigma(z) = \sum_{i=1}^{k} y_i(z_{j_i}),$$

where, for each i,

$$y_i = \mathrm{Ad}(\sigma_{j_1} \cdots \sigma_{j_{i-1}}) \in \mathrm{Aut}(\mathbf{g}) \quad \text{if } \varepsilon_i = 1,$$

or

$$y_i = -\mathrm{Ad}(\sigma_{j_1} \cdots \sigma_{j_i}) \in \mathrm{Aut}(\mathbf{g}) \quad \text{if } \varepsilon_i = -1.$$

Proof. Let $v = (v_1, v_2, \dots, v_n) \in T_\sigma((S^3)^n) = T_{\sigma_1} S^3 \times \cdots \times T_{\sigma_n} S^3$. We assume the basic facts (true in any Lie group)

(i) $(dq_i)_\sigma(v) = v_i$, and
(ii) $(dq_i^{-1})_\sigma(v) = -\sigma_i^{-1} v_i \sigma_i^{-1}$.

By the product rule,

$$dw_\sigma(v) = \sum_{i=1}^{k} \sigma_{j_1}^{\varepsilon_1} \cdots \sigma_{j_{i-1}}^{\varepsilon_{i-1}} (dq_{j_i}^{\varepsilon_i})_\sigma(v) \sigma_{j_{i+1}}^{\varepsilon_{i+1}} \cdots \sigma_{j_k}^{\varepsilon_k}.$$

We then compute

$$(*) \qquad \begin{aligned} d\tilde{w}_\sigma(z) &= (dw_\sigma(z\sigma)) w(\sigma)^{-1} \\ &= \sum_{i=1}^{k} \sigma_{j_1}^{\varepsilon_1} \cdots \sigma_{j_{i-1}}^{\varepsilon_{i-1}} (dq_{j_i}^{\varepsilon_i})_\sigma(z\sigma) \sigma_{j_{i+1}}^{\varepsilon_{i+1}} \cdots \sigma_{j_k}^{\varepsilon_k} w(\sigma)^{-1}. \end{aligned}$$

Using facts (i) and (ii), and the fact that

$$w(\sigma)^{-1} = \sigma_{j_k}^{-\varepsilon_k} \cdots \sigma_{j_1}^{-\varepsilon_1},$$

the expression $(*)$ simplifies to the statement in the lemma. □

Suppose $\rho \in R(\Gamma)$, and $w = x_{j_1}^{\varepsilon_1} \cdots x_{j_k}^{\varepsilon_k}$ is one of the relations in our presentation of Γ. Then the coordinates of ρ in $(S^3)^n$ are $(\rho(x_1), \dots, \rho(x_n))$. The following proposition, which is an immediate consequence of Lemma 17, enables us to compute $T_\rho(R(\Gamma))$.

Proposition 18. *Let* $z = (z_1, z_2, \dots, z_n)$ *be an element of* $\mathbf{g}^n$. *Then*

$$dw_\rho(z) = \sum_{i=1}^{k} u_i(z_{j_i}),$$

where, for each i,

$$u_i = \tilde{\rho}(x_{j_1}^{\varepsilon_1} \cdots x_{j_{i-1}}^{\varepsilon_{i-1}}) \in \mathrm{Aut}(\mathbf{g}) \quad \text{if } \varepsilon_i = 1,$$

or

$$u_i = -\tilde{\rho}(x_{j_1}^{\varepsilon_1} \cdots x_{j_1}^{\varepsilon_i}) \in \mathrm{Aut}(\mathbf{g}) \quad \text{if } \varepsilon_i = -1.$$

This proposition enables us to express $T_\rho R(\Gamma)$ as the space of solutions to a specific system of linear equations, i.e., as the null space of a specific matrix.

Note. As Weil [W] and others have observed, this is the same system of equations that defines the space of 1-cocycles of Γ with coefficients in g, which is viewed as a Γ-module via $\tilde{\rho}$.

II.B. The tangent space at an abelian representation. Let $K \subset S^3$ be a knot. Recall that $S(K)$ denotes the space of abelian representations. The main result of this section is the following theorem.

Theorem 19. *Let* $\rho \in R(K)$ *be an abelian representation. Let* $\mu \in \pi_1(S^3 - K)$ *be a meridian of* K. *Assume (by conjugation in* S^3*) that* $\rho(\mu) \in \mathbb{C}$. *Let* $\Delta_K(t)$ *be the Alexander polynomial of* K. *Then*:

(i) *If* $\Delta_K(\rho(\mu)^2) \neq 0$, *then* $\dim T_\rho(R(K)) = 3$ *and a sufficiently small neighborhood of* ρ *in* $R(K)$ *consists entirely of points of the* 3*-manifold* $S(K)$.

(ii) *If* $\Delta_K(\rho(\mu)^2) = 0$, *then*

$$\dim T_\rho(R(K)) = 3 + 2\dim_{\mathbb{C}}(\ker A_K(\rho(\mu)^2)),$$

where $A_K(t)$ *is obtained from an Alexander matrix of* K *by deleting a column, and has the property that* $\Delta_K(t) = \det(A_K(t))$.

Note. This theorem has been discovered independently by Steve Boyer jointly with Andrew Nicas, and also by Charles Frohman. However, we know of no reference for it in the literature.

Proof of Theorem 19. Let $\Gamma = \pi_1(S^3 - K)$. Then Γ has a Wirtinger presentation of the form

$$\Gamma = \{x_1, \ldots, x_n : w_1, \ldots, w_{n-1}\}.$$

The relators w_i are words in the x_j of the form

$$w_i = (x_{j_i})^{\varepsilon_i} x_i (x_{j_i})^{-\varepsilon_i} (x_{i+1})^{-1},$$

where $\varepsilon_i = \pm 1$ and $j_i \in \{1, 2, \ldots, n\}$ for each $i = 1, \ldots, n-1$. Let $W: (S^3)^n \to (S^3)^{n-1}$ be the function whose ith component is w_i. Then

$$T_\rho(R(K)) = \ker(dW_\rho : \mathbf{g}^n \to \mathbf{g}^{n-1}) = \bigcap \{\ker(dw_i)_\rho : i = 1, \ldots, n\}.$$

Using Proposition 18, we now calculate $(dw_i)_\rho$ as follows: If $\varepsilon_i = 1$, then

$$\begin{aligned}(dw_i)_\rho&(z_1, \ldots, z_n)\\ &= z_{j_i} + \tilde\rho(x_{j_1}) z_i - \tilde\rho(x_{j_i} x_i x_{j_i}^{-1}) z_{j_i} - \tilde\rho(x_{j_i} x_i x_{j_i}^{-1} (x_{i+1})^{-1}) z_{i+1}\\ &= (\text{using the relations for } \Gamma)\, \tilde\rho(x_{j_i}) z_i - z_{i+1} + (I - \tilde\rho(x_{i+1})) z_{j_i},\end{aligned}$$

where $(z_1, \ldots, z_n) \in \mathbf{g}^n$. If $\varepsilon_i = -1$, then we obtain

$$(dw_i)_\rho(z_1, \ldots, z_n) = \tilde\rho(x_{j_i}^{-1}) z_i - z_{i+1} + (\tilde\rho(x_{j_i}^{-1} x_i) - \tilde\rho(x_{j_i}^{-1})) z_{j_i}.$$

Since ρ is abelian we may assume, by conjugation, that

$$\rho(x_1) = \cdots = \rho(x_n) = e^{\mathrm{i}\theta}.$$

Using $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as a basis for $\mathbf{g}$, an easy computation shows that

$$\tilde\rho(x_1) = \cdots = \tilde\rho(x_n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}.$$

We denote this matrix by T.

$dW_\rho : \mathbf{g}^n \to \mathbf{g}^{n-1}$ is described by an $(n-1) \times n$ matrix A_ρ with entries in $\mathrm{GL}(\mathbf{g}) = \mathrm{GL}(3, \mathbb{R})$, the ith row of A_ρ being $(dw_i)_\rho$. Thus, if $\varepsilon_i = 1$, the ith row of A_ρ is given by

$$\begin{array}{rccccc} \text{column number}: & i & i+1 & \cdots & j_i \\ \text{entry}: & T & -1 & \cdots & I - T \end{array}$$

with all other entries in this row equal to zero. If $\varepsilon_i = -1$, the ith row of A_ρ is given by

$$\begin{array}{rccccc} \text{column number}: & i & i+1 & \cdots & j_i \\ \text{entry}: & T^{-1} & -I & \cdots & I - T^{-1} \end{array}$$

with all other entries equal to zero. Note that A_ρ is just the Alexander matrix, as computed from the same projection as our original Wirtinger presentation, only with the real number 1 and the variable t replaced by the 3×3 matrices I and T.

Since the sum of the columns of A_ρ is zero, the kernel of dW_ρ contains all vectors of the form $(z, \ldots, z) \in \mathbf{g}^n$, i.e., the diagonal $\Delta(\mathbf{g}^n)$. This reflects the fact that

$$\Delta(\mathbf{g}^n) = T_\rho(\Delta(S^3)^n),$$

while

$$\Delta(S^3)^n = S(K) \subset R(K)$$

is precisely the set of abelian representations.

Let us now examine $\ker(A_\rho)/\Delta(\mathbf{g}^n)$. Define

$$L_\rho = \{(z_1, \ldots, z_{n-1}, 0) \in \ker(A_\rho)\}.$$

We then have $L_\rho \cong \ker(A_\rho)/\Delta(\mathbf{g}^n)$ by the inclusion composed with the quotient

$$L_\rho \to \ker(A_\rho) \to \ker(A_\rho)/\Delta(\mathbf{g}^n).$$

Observe that L_ρ is just the kernel of the matrix obtained by removing the last column from A_ρ. This matrix, which we call $\widetilde{A}_\rho$, is an $(n-1)\times(n-1)$ matrix with entries in $\mathrm{GL}(3, \mathbb{R})$, or a $3(n-1) \times 3(n-1)$ matrix with entries in $\mathbb{R}$. If $i < n-1$ and $\varepsilon_i = 1$, the ith row of $\widetilde{A}_\rho$ becomes:

$$\begin{array}{ccc|ccc|ccc}
\text{column}: & i & & & i+1 & & & j_i & \\
 & & & & & & & & \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & \cos 2\theta & -\sin 2\theta & 0 & -1 & 0 & 0 & 1-\cos 2\theta & \sin 2\theta \\
0 & \sin 2\theta & \cos 2\theta & 0 & 0 & -1 & 0 & -\sin 2\theta & 1-\cos 2\theta
\end{array}$$

The other entries in this row are equal to zero. If $\varepsilon_1 = -1$, the ith row is similar, and the following discussion applies to both. The entry in column j_i is, of course, omitted if $j_i = n$. Row $n-1$ is similar, but without the entry in column $i+1$.

Suppose the null space of $\widetilde{A}_\rho$ contains a vector with transpose $(a_1, b_1, c_1, \ldots, a_{n-1}, b_{n-1}, c_{n-1})$. (We are writing this vector as a vector in $\mathbb{R}^{3(n-1)} \cong g^{n-1}$ using the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for each copy of $\mathbf{g}$.) Row i of $\widetilde{A}_\rho$, for $i < n-1$, tells us that $a_i = a_{i+1}$. Row $n-1$ tells us that $a_{n-1} = 0$. So we know that

$$a_1 = \cdots = a_{n-1} = 0.$$

Now consider the $(n-1)\times(n-1)$ matrix over $\mathbb{C}$ obtained from $\widetilde{A}_\rho$ by replacing I by 1 and T by $\cos 2\theta + \mathbf{i}\sin 2\theta$, in each of the nonzero entries $-I$, $I-T$, etc. Denote this matrix by $\widetilde{A}'$. Note that $\widetilde{A}'$ is just the Alexander matrix with $e^{2\theta\mathbf{i}}$ substituted for t, and the last column removed. It is immediate that the vector with transpose $(0, b_1, c_1, 0, b_2, c_2, \ldots, 0, b_{n-1}, c_{n-1})$ is in the null space of $\widetilde{A}_\rho$ if and only if the vector with transpose

$$(b_1 + \mathbf{i}c_1, \ldots, b_{n-1} + \mathbf{i}c_{n-1})$$

is in the nullspace of $\widetilde{A}'$. Note that the latter nullspace is a vector space over $\mathbb{C}$. It follows that

$$\dim_{\mathbb{R}} L_\rho = \dim_{\mathbb{R}}(\ker(\tilde{A}_\rho)) = 2\dim_{\mathbb{C}}(\ker \widetilde{A}').$$

Since $\widetilde{A}' = A_K(\rho(\mu)^2)$, it follows that

$$\dim T_\rho(R(K)) = 3 + \dim L_\rho = 3 + 2\dim_{\mathbb{C}}(\ker A_K(\rho(\mu)^2)),$$

which proves part (ii) of the theorem.

If $\Delta_K(\rho(\mu)^2) \neq 0$, then $\widetilde{A}'$ is nonsingular, so $\dim T_\rho(R(K)) = 3$. Since this means that dW_ρ has maximal rank, it follows that $R(K)$ is a 3-manifold in a small neighborhood of ρ. Since $S(K) \subset R(K)$ is a 3-manifold containing ρ, this small neighborhood must consist only of abelian representations. This proves part (i) of the theorem. □

In the cases of the torus and twist knots analyzed in §I of this thesis, those abelian representations ρ for which $\dim T_\rho R(K) > 3$, i.e., those for which $\Delta_K(\rho(\mu)^2) = 0$, are precisely those abelian representations which can be expressed as limits of arcs of nonabelian representations. The above theorem implies that, for an arbitrary knot, abelian representations ρ satisfying $\Delta_K(\rho(\mu)^2) = 0$ are the only abelian representations which could possibly be expressible as limits of nonabelian representations. The question of whether these singular (i.e., $\Delta_K(\rho(\mu)^2) = 0$) abelian representations can always be expressed as limits of nonabelian representations is a question of when infinitesimal deformations can be realized by actual deformations. It has now been proven (see [FK]) that if ρ is an abelian representation and $\rho(\mu)^2$ is a simple root of Δ_K, then ρ is a limit of an arc of nonabelian representations.

$T_\rho(R(K))$ and Euclidean isometries. We will now give an interpretation of the tangent space at an abelian representation in terms of Euclidean representations of Γ. Define $\mathrm{Isom}_+(\mathbb{C})$ to be the set of orientation-preserving Euclidean isometries of $\mathbb{C}$. These are the maps of the form $u \to au + b$, where a and b are elements of $\mathbb{C}$ and $|a| = 1$. The conjugacy classes in this group are of three types: $\{\mathrm{Id}_{\mathbb{C}}\}$, X_r (r a positive real), and Y_a ($a \in S^1$, $a \neq 1$), where

$$X_r = \{u \to u + b : |b| = r\}$$

and

$$Y_a = \{u \to au + b : b \in \mathbb{C}\}.$$

Consider a homomorphism

$$\varphi : \Gamma \to \mathrm{Isom}_+(\mathbb{C}).$$

Since the generators x_i are conjugate to each other, their images

$$\varphi(x_i)(u) = au + (b_i + \mathbf{i}c_i)$$

all have the same multiplicative factor a. Fixing a, we consider the set of representations of the form $\varphi(x_i) = (u \to au + (b_i + \mathbf{i}c_i))$. Suppose, in addition, we impose the condition that $\varphi(x_n)$ fixes the origin, i.e., that $b_n + \mathbf{i}c_n = 0$. De Rham [D, pp. 188–189] shows that the set of $(n-1)$-tuples

$$(b_1 + \mathbf{i}c_1, b_2 + \mathbf{i}c_2, \dots, b_{n-1} + \mathbf{i}c_{n-1})$$

making φ a homomorphism is precisely the null-space of the matrix $A_K(a)$. (As above, $A_K(t)$ is obtained by removing the nth column from the Alexander matrix associated with the Wirtinger presentation of Γ used in the above proof.) It follows that we can identify these Euclidean representations with certain tangent vectors.

To make this precise, let $\rho \in R(K)$ be an abelian representation. Assume, by conjugation, that $\rho(\mu) \in \mathbb{C}$. Recall the subspace $L_\rho \subset T_\rho(R(K))$ defined in the last proof. Define a map

$$f_\rho : L_\rho \to \mathrm{Hom}(\Gamma, \mathrm{Isom}_+(\mathbb{C}))$$

given by

$$(0, b_1, c_1, \dots, 0, b_{n-1}, c_{n-1}, 0, 0, 0) \to \varphi,$$

where $\varphi(x_i)(u) = \rho(\mu)^2 u + (b_i + \mathbf{i}c_i)$. De Rham's result with $a = \rho(\mu)^2$ combines with the proof of Theorem 19 to prove the following proposition.

Proposition 20. *f_ρ induces an isomorphism $L_\rho \cong \{\varphi \in \mathrm{Hom}(\Gamma, \mathrm{Isom}_+(\mathbb{C})) : \varphi$ has multiplicative factor equal to $\rho(\mu)^2$, and $\varphi(x_n)$ fixes the origin in $\mathbb{C}\}$.* □

To conclude, we give a geometric interpretation of this isomorphism. Recall from §I.C that one can associate to a representation of Γ in S^3 a geodesic image in S^2 of a certain polygon P_K, whose vertices correspond to the generators of Γ. This image must also satisfy certain geometric constraints imposed by the relations between the x_i. Let $v \in L_\rho$ be a Zariski tangent vector which is actually tangent to an arc of representations, say ρ_s. We are assuming that $\rho_0 = \rho$ is abelian, while the other ρ_s are not necessarily abelian. As s approaches 0, the vertices of the corresponding images of P_K in S^2 approach a single point. But as the diameter of the image of P_K in S^2 approaches 0, that image more and more closely approximates a Euclidean image of P_K. After multiplying by an appropriate scaling factor we can see that the spherical images of P_K actually approach a Euclidean image satisfying the same geometric constraints. This Euclidean image, in turn, corresponds to a representation of Γ in $\mathrm{Isom}_+(\mathbb{C})$, by letting the images of the vertices correspond to the centers of rotation of the images of the generators of Γ. This gives us a geometric interpretation of the map f_ρ, for those vectors which are realizable by arcs of representations.

Let $\rho \in R(K)$ be an abelian representation with $\rho(\mu) \in \mathbb{C}$. A corollary of this discussion is the following:

Proposition 21. $\dim T_\rho(R(K)) > 3$ *if and only if there exists a nonabelian representation of Γ in $\mathrm{Isom}_+(\mathbb{C})$ which takes the generators x_i to rotations of angle* $\arg(\rho(\mu)^2)$. □

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