

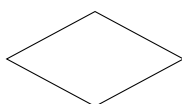
1. Let  $G$  be a group. Show that a nonempty subset  $H \subset G$  is a subgroup if and only if the following holds: for all  $a, b \in H$ ,  $ab^{-1} \in H$ .
2. Let  $G$  be any group. Let  $a, b \in G$  and let  $x \in G$  be some unknown. Suppose

$$x^3 = e, \quad x^2b = ba.$$

- (a) Solve for  $x$  in terms of  $a$  and  $b$ . Show all steps.
  - (b) Suppose  $G$  is  $(\mathbb{Z}_5^\times, \times)$  and  $a \equiv 4 \pmod{5}$ ,  $b \equiv 3 \pmod{5}$ . Find  $x$ .
3. Prove that the symmetric group  $S_n$  is nonabelian if and only if  $n \geq 3$ .
  4. Consider the following set of real  $3 \times 3$  matrices:

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

- (a) Show that  $G$  with the operation of matrix multiplication is a group.
  - (b) Is the group  $G$  abelian? Explain.
5. List the left and right cosets of the subgroups in each of the following.
    - (a)  $3\mathbb{Z}$  in  $\mathbb{Z}$
    - (b)  $A_4$  in  $S_4$
    - (c)  $\langle 8 \rangle$  in  $\mathbb{Z}_{24}$
    - (d)  $H = \{e, (123), (132)\}$  in  $S_4$
  6. Consider the following diamond shape situated inside the 2-dimensional plane:



- (a) Draw all the symmetries of this shape. How big is the group of symmetries?
  - (b) Label the vertices with numbers 1, 2, 3, 4. Write down the permutation of vertices corresponding to each symmetry.
  - (c) Is the corresponding permutation group a subgroup of an alternating group?
7. Use Fermat's Little Theorem to show that if  $p = 4n + 3$  is prime, then there is no solution to the equation  $x^2 \equiv -1 \pmod{p}$ .
  8.
    - (a) Consider  $(142)(231)$  and  $(54123)(24)$ . Compute each as a product of disjoint cycles. What are the orders of these elements, and what are their parities?
    - (b) Is  $\{e, (12), (34), (12)(34), (45), (12)(45)\} \subset S_5$  a subgroup?
    - (c) What are the possible orders of elements in the alternating group  $A_5$ ?

9. Compute  $7^{81} \pmod{30}$ .
10. Let  $G$  be a finite cyclic group of order  $n$  generated by  $a$ . Show that if  $b = a^k$  where  $k$  is relatively prime to  $n$ , then  $b$  generates  $G$ .