

1. Let G be a group. Show that a nonempty subset $H \subset G$ is a subgroup if and only if the following holds: for all $a, b \in H$, $ab^{-1} \in H$.

2. Let G be any group. Let $a, b \in G$ and let $x \in G$ be some unknown. Suppose

$$x^3 = e, \quad x^2b = ba.$$

(a) Solve for x in terms of a and b . Show all steps.
 (b) Suppose G is $(\mathbb{Z}_5^\times, \times)$ and $a \equiv 4 \pmod{5}$, $b \equiv 3 \pmod{5}$. Find x .

3. Prove that the symmetric group S_n is nonabelian if and only if $n \geq 3$.

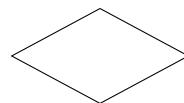
4. Consider the following set of real 3×3 matrices:

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

(a) Show that G with the operation of matrix multiplication is a group.
 (b) Is the group G abelian? Explain.
 5. List the left and right cosets of the subgroups in each of the following.

(a) $3\mathbb{Z}$ in \mathbb{Z}
 (b) A_4 in S_4
 (c) $\langle 8 \rangle$ in \mathbb{Z}_{24}
 (d) $H = \{e, (123), (132)\}$ in S_4

6. Consider the following diamond shape situated inside the 2-dimensional plane:



(a) Draw all the symmetries of this shape. How big is the group of symmetries?
 (b) Label the vertices with numbers 1, 2, 3, 4. Write down the permutation of vertices corresponding to each symmetry.
 (c) Is the corresponding permutation group a subgroup of an alternating group?
 7. Use Fermat's Little Theorem to show that if $p = 4n + 3$ is prime, then there is no solution to the equation $x^2 \equiv -1 \pmod{p}$.
 8. (a) Consider $(142)(231)$ and $(54123)(24)$. Compute each as a product of disjoint cycles. What are the orders of these elements, and what are their parities?
 (b) Is $\{e, (12), (34), (12)(34), (45), (12)(45)\} \subset S_5$ a subgroup?
 (c) What are the possible orders of elements in the alternating group A_5 ?

9. Compute $7^{81} \pmod{30}$.
10. Let G be a finite cyclic group of order n generated by a . Show that if $b = a^k$ where k is relatively prime to n , then b generates G .