Alternating groups

In this lecture we continue studying even and odd permutations. We introduce and study the *alternating groups* A_n which consist of *even* permutations. We then consider the rotational symmetries of the tetrahedron, which are closely related the group A_4.

Recall that a permutation $\sigma \in S_n$ is *even* if $\sigma(\Delta_n) = \Delta_n$ and *odd* if $\sigma(\Delta_n) = -\Delta_n$. Here Δ_n is the polynomial $\prod_{1 \leqslant i<j \leqslant n}(x_i - x_j)$ introduced last lecture. Define

$$A_n = \{\sigma \in S_n : \sigma \text{ is even}\} \subset S_n$$

▸ **The subset $A_n \subset S_n$ is a subgroup, called the n^{th} *alternating group*.**

To prove this we first record a useful relation. Given any permutations $\sigma, \sigma' \in S_n$ we have

$$(\sigma\sigma')(\Delta_n) = \sigma(\sigma'(\Delta_n))$$

This just follows by writing out what each side means explicitly:

$$\begin{aligned}(\sigma\sigma')(\Delta_n) &= \prod_{1 \leqslant i<j \leqslant n} (x_{(\sigma\sigma')(i)} - x_{(\sigma\sigma')(j)}) = \prod_{1 \leqslant i<j \leqslant n} (x_{\sigma(\sigma'(i))} - x_{\sigma(\sigma'(j))}) \\ &= \sigma\left(\prod_{1 \leqslant i<j \leqslant n} (x_{\sigma'(i)} - x_{\sigma'(j)})\right) = \sigma(\sigma'(\Delta_n))\end{aligned}$$

Similarly, $\sigma(-\Delta_n) = -\sigma(\Delta_n)$. Now suppose $\sigma(\Delta_n) = (-1)^k\Delta_n$ and $\sigma'(\Delta_n) = (-1)^l\Delta_n$. Then

$$\begin{aligned}(\sigma\sigma')(\Delta_n) &= \sigma(\sigma'(\Delta_n)) = \sigma((-1)^l\Delta_n) \\ &= (-1)^l\sigma(\Delta_n) = (-1)^l(-1)^k\Delta_n \\ &= (-1)^{l+k}\Delta_n\end{aligned}$$

From this computation we see the same rules as for adding even and odd integers:

σ	σ'	$\sigma\sigma'$
even	even	even
odd	even	odd
even	odd	odd
odd	odd	odd

In particular, if $\sigma, \sigma' \in A_n$ (σ, σ' are both even) then $\sigma\sigma' \in A_n$ ($\sigma\sigma'$ is even). Also the identity is even, so it is in A_n. Further, if $\sigma \in A_n$ (σ is even), then since $\sigma\sigma^{-1} = e \in A_n$ ($\sigma\sigma^{-1}$ is even) we must have $\sigma^{-1} \in A_n$ (σ^{-1} is even). Thus A_n is a subgroup of S_n.

An alternative characterization of the parity of a permutation is as follows:

▸ **If $\sigma = \tau_1 \cdots \tau_k$ where τ_i are transpositions, then σ is odd if and only if k is odd.**

To prove this we first show that every transposition is odd. First consider the transposition $\sigma = (12) \in S_n$. There are four kinds of factors in Δ_n:

$$(x_1 - x_2), \qquad (x_1 - x_j)\ \ (j > 2), \qquad (x_2 - x_j)\ \ (j > 2), \qquad (x_i - x_j)\ \ (j > i > 2)$$

Now $\sigma = (12)$ only swaps 1 and 2. So it sends the first type of factor to its negative $(x_2 - x_1) = -(x_2 - x_1)$. It interchanges the second and third types (preserving signs), and fixes all factors of the fourth type. Taking the product we conclude $\sigma(\Delta_n) = -\Delta_n$, where the sign comes from the effect of $\sigma = (12)$ on the factor $(x_1 - x_2)$. Next, we use:

▸ **Let $\sigma = (a_1\, a_2 \cdots a_k) \in S_n$ be a cycle and $\tau \in S_n$ any other permutation. Then**

$$\tau\sigma\tau^{-1} = (\tau(a_1)\ \ \tau(a_2)\ \ \cdots\ \ \tau(a_k))$$

A special case is when $\sigma = (i\, j)$ a transposition different from (12) with $j > i$. Setting $\tau = (i\ 1)(j\ 2)$ we get $\tau^{-1}\sigma\tau = (12)$. If $i = 1$, interpret $(i\ 1)$ as e.

Now let σ be any transposition and choose τ as above such that $\tau\sigma\tau^{-1} = (12)$. Then $\sigma = \tau^{-1}(12)\tau$. Let $\tau(\Delta_n) = (-1)^k\Delta_n$. Note also $\tau^{-1}(\Delta_n) = (-1)^k\Delta_n$. We then compute

$$\begin{aligned}
\sigma(\Delta_n) &= (\tau^{-1}(12)\tau)(\Delta_n) \\
&= \tau^{-1}((12)(\tau(\Delta_n))) \\
&= \tau^{-1}((12)(-1)^k\Delta_n)) \\
&= (-1)^k\tau^{-1}((12)(\Delta_n)) \\
&= (-1)^k\tau^{-1}(-\Delta_n) \\
&= (-1)^{k+1}\tau^{-1}(\Delta_n) \\
&= (-1)^{2k+1}\Delta_n = -\Delta_n
\end{aligned}$$

This completes our claim that every transposition is odd. Then to prove the claim about $\sigma = \tau_1 \cdots \tau_k$ for a product of transpositions, we use the rules of the table we determined above.

Let us look at some examples. As $S_1 = \{e\}$ we of course have $A_1 = \{e\}$. Next, $S_2 = \{e, (12)\}$, and (12) is odd, so in fact $A_2 = \{e\}$ as well. The 3rd symmetric group is

$$S_3 = \{e, (12), (23), (31), (123), (132)\}$$

The three transpositions (12), (23), (31) are odd, so they are not in A_3. On the other hand $(123) = (13)(12)$ and $(132) = (12)(13)$, so these are even. Thus

$$A_3 = \{e, (123), (132)\}$$

Note that $(123)^2 = (132)$ and $(123)^3 = e$, so A_3 is a cyclic (hence abelian) group of order 3. This is in contrast to S_3. However:

▸ **The alternating group A_n is non-abelian if and only if $n \geqslant 4$.**

Symmetries of the tetrahedron

The first non-abelian alternating group, A_4, is closely related to the rotational symmetries of the tetrahedron in 3-dimensional space.

A tetrahedron is a solid in 3-dimensional Euclidean space which has 4 vertices and 4 sides, each an equilateral triangle. On the next page we list the symmetries of the tetrahedron. There are 2 types of non-identity symmetries. The first type $(R_1^{\pm 1}, R_2^{\pm 1}, R_3^{\pm 1}, R_4^{\pm 1})$ fixes a vertex and rotates the tetrahedron around an axis passing through the fixed vertex by 120° in one of two directions. The second kind of symmetry (A, B, C) is a 180° rotation through an axis which passes through the centers of two opposite edges.

If we label the vertices of the tetrahedron by $\{1, 2, 3, 4\}$ we can associate a permutation to each symmetry. Magically, the subgroup of S_4 corresponding to the symmetries of the tetrahedron is A_4! Below we include the Cayley table.

	e	R_1	R_1^{-1}	R_2	R_2^{-1}	R_3	R_3^{-1}	R_4	R_4^{-1}	A	B	C
e	e	R_1	R_1^{-1}	R_2	R_2^{-1}	R_3	R_3^{-1}	R_4	R_4^{-1}	A	B	C
R_1	R_1	R_1^{-1}	e	A	R_4	B	R_2	C	R_3	R_3^{-1}	R_4^{-1}	R_2^{-1}
R_1^{-1}	R_1^{-1}	e	R_1	R_3^{-1}	C	R_4^{-1}	A	R_2^{-1}	B	R_2	R_3	R_4
R_2	R_2	C	R_4^{-1}	R_2^{-1}	e	R_1	B	R_3^{-1}	A	R_1^{-1}	R_4	R_3
R_2^{-1}	R_2^{-1}	R_3	A	e	R_2	C	R_4	B	R_1^{-1}	R_4^{-1}	R_3^{-1}	R_1
R_3	R_3	A	R_2^{-1}	R_4^{-1}	B	R_3^{-1}	e	R_1	C	R_4	R_1^{-1}	R_2
R_3^{-1}	R_3^{-1}	R_4	B	C	R_1^{-1}	e	R_3	A	R_2	R_1	R_2^{-1}	R_4^{-1}
R_4	R_4	B	R_3^{-1}	R_1	A	R_2^{-1}	C	R_4^{-1}	e	R_3	R_2	R_1^{-1}
R_4^{-1}	R_4^{-1}	R_2	C	B	R_3	A	R_1^{-1}	e	R_4	R_2^{-1}	R_1	R_3^{-1}
A	A	R_2^{-1}	R_3	R_4	R_1	R_1^{-1}	R_4^{-1}	R_2	R_3^{-1}	e	C	B
B	B	R_3^{-1}	R_4	R_3	R_4^{-1}	R_2	R_1	R_1^{-1}	R_2^{-1}	C	e	A
C	C	R_4^{-1}	R_2	R_1^{-1}	R_3^{-1}	R_4	R_2^{-1}	R_3	R_1	B	A	e

Symmetries of the tetrahedron.