

Homework 7

1. Consider the following group, which you may think of as $GL_2(\mathbb{Z}_2)$:

$$G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_2, ad - bc \equiv 1 \pmod{2} \right\}$$

The operation is matrix multiplication, although we now use addition and multiplication in \mathbb{Z}_2 instead of \mathbb{R} . You may verify that this is a group, although you do not need to prove it on your homework. Show that G is isomorphic to the group S_3 .

2. In class we applied the construction of Cayley's Theorem to the quaternion group Q_8 to produce a 1-1 homomorphism $\phi: Q_8 \rightarrow S_8$. Apply the same construction to the following groups, to get homomorphisms into symmetric groups:

- (a) $(\mathbb{Z}_3, +)$
- (b) $(\mathbb{Z}_8^\times, \times)$
- (c) S_3

3. Consider the quaternion group $Q_8 = \{1, -1, \mathbf{i}, -\mathbf{i}, \mathbf{j}, -\mathbf{j}, \mathbf{k}, -\mathbf{k}\}$.

- (a) List all subgroups of Q_8 .
- (b) Show that every subgroup of Q_8 is normal.
- (c) Construct an onto homomorphism $Q_8 \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$. From your homomorphism, what does the 1st Isomorphism Theorem imply?

4. Determine which of the following are rings, with the usual operations of addition and multiplication. If it is a ring you do not need to write out all the axioms, but if it is not a ring, explain why. If the set is a ring, is it a field? If not, explain why.

- (a) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$
- (b) $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, where $i = \sqrt{-1}$, so that $i^2 = -1$
- (c) The set of following matrices *with entries in the ring \mathbb{Z}_2* :

$$\left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

- (d) The set of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$. Addition and multiplication is defined pointwise. That is, for two such functions f, g , we have $f + g$ and fg defined by $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$ for $x \in \mathbb{R}$.

5. For each ring R given, determine the set of units R^\times .

- (a) The integers \mathbb{Z}
- (b) The ring $M_2(\mathbb{Z})$, the 2×2 matrices with integer entries
- (c) Each of the rings in problem #4