# Homework 4

1. List the left and right cosets of the subgroups in the following list.
   (a) The subgroup $\langle 5 \rangle$, generated by 5 (mod 20), inside $(\mathbb{Z}_{20}, +)$.
   (b) The subgroup $4\mathbb{Z} = \{4k : k \in \mathbb{Z}\}$ inside the group $(\mathbb{Z}, +)$.
   (c) The subgroup $A_3$ inside the symmetric group $S_3$.
   (d) The subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ in the group $A_4$.
   (e) The subgroup $H = \{e, (123), (132)\}$ in the group $A_4$.

   For which of these examples does it happen that every right coset is a left coset, and every left coset is a right coset?

2. Let $G$ be a group and $H \subset G$ a subgroup with index 2, i.e. $[G : H] = 2$. Show that $aH = Ha$ for all $a \in G$.

3. Recall that $\mathrm{GL}_2(\mathbb{R})$ is the group of real $2 \times 2$ matrices with non-zero determinant, and $\mathrm{SL}_2(\mathbb{R})$ is the subgroup of those matrices with determinant 1. Describe the right cosets of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{GL}_2(\mathbb{R})$, and find the index of this subgroup.

4. Use Euler's Theorem or Fermat's Little Theorem to help compute the following.
   (a) $7^{26}$ (mod 15)
   (b) The last digit of $97^{123}$ (Hint: pass to integers mod 10)
   (c) $15^{83}$ (mod 41)

5. Suppose $G$ is a finite group, and $a \in G$. Suppose $n$ is an integer greater than 1 that divides the order of $G$. Show that $a^n$ cannot generate $G$, i.e. $\langle a^n \rangle \neq G$.

6. Let $G$ be a finite group of order $pq$ where $p$ and $q$ are distinct primes. Show that if $a, b \in G$ are non-identity elements of different orders, then the only subgroup in $G$ containing $a$ and $b$ is the whole group $G$.

7. Let $G$ be a group. Given $a, b \in G$, we say $a$ is *conjugate* to $b$ if there exists $g \in G$ such that $a = gbg^{-1}$. Define $\sim$ as follows: $a \sim b$ if and only if $a$ is conjugate to $b$.
   (a) Show that $\sim$ is an equivalence relation on $G$.
   (b) What are the equivalence classes of this relation if $G$ is abelian?
   (c) Compute the equivalence classes of this relation for the groups $S_3$ and $A_4$.