Math 310 **Final Exam**
Spring 2024 **Wed May 8**

No electronic devices are allowed. No collaboration is allowed. There are 10 pages and each page is worth 6 points, for a total of 60 points.

1. Dot Product.

(a) Find one non-zero vector in $\mathbb{R}^3$ that is perpendicular to the vector $\langle 1, -1, 3\rangle$. [There are infinitely many correct answers.]

We need to find a non-zero vector $\langle a, b, c\rangle$ satisfying

$$\begin{aligned} \langle a, b, c\rangle \bullet \langle 1, -1, 3\rangle &= 0 \\ a - b + 3c &= 0. \end{aligned}$$

For example, take $\langle a, b, c\rangle = \langle 1, 1, 0\rangle$.

(b) Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors satisfying $\mathbf{u} \bullet \mathbf{v} = 3$, $\mathbf{u} \bullet \mathbf{u} = 2$ and $\mathbf{v} \bullet \mathbf{v} = 9$. Compute $\cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, measured tail-to-tail.

The dot product theorem gives

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{\mathbf{u} \bullet \mathbf{v}}{\sqrt{\mathbf{u} \bullet \mathbf{u}}\sqrt{\mathbf{v} \bullet \mathbf{v}}} = \frac{3}{\sqrt{2}\sqrt{9}} = \frac{1}{\sqrt{2}}.$$

[Remark: Hence $\theta = \pi/4$. I could have asked for θ but I know that some students don't remember that $\cos(\pi/4) = 1/\sqrt{2}$, and this isn't a trigonometry class. Some people would say that I should have asked for it, precisely **because** some of the students don't know it. But this is just Problem 1(b).]

2. Cross Product.

(a) Find one non-zero vector in $\mathbb{R}^3$ that is perpendicular to both $\langle 2, -1, 0\rangle$ and $\langle 2, 1, -1\rangle$. [There are infinitely many correct answers.]

The cross product is designed to satisfy this property:

$$\begin{aligned} \langle 2, -1, 0\rangle \times \langle 2, 1, -1\rangle &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix} \\ &= \langle (-1)(-1) - (1)(0), (2)(0) - (2)(-1), (2)(1) - (2)(-1)\rangle \\ &= \langle 1, 2, 4\rangle. \end{aligned}$$

Check:

$$\langle 1, 2, 4\rangle \bullet \langle 2, -1, 0\rangle = 0 \quad \text{and} \quad \langle 1, 2, 4\rangle \bullet \langle 2, 1, -1\rangle = 0.$$

(b) Find the equation of the plane in $\mathbb{R}^3$ that contains the three points $(0,0,0)$, $(2,-1,0)$ and $(2,1,-1)$.

The plane contains the point $(0,0,0)$ and has normal vector $\langle 1,2,4\rangle$. Hence the equation of the plane is

$$\begin{aligned} \langle 1,2,4\rangle \bullet \langle x-0, y-0, z-0\rangle &= 0 \\ x+2y+4z &= 0. \end{aligned}$$

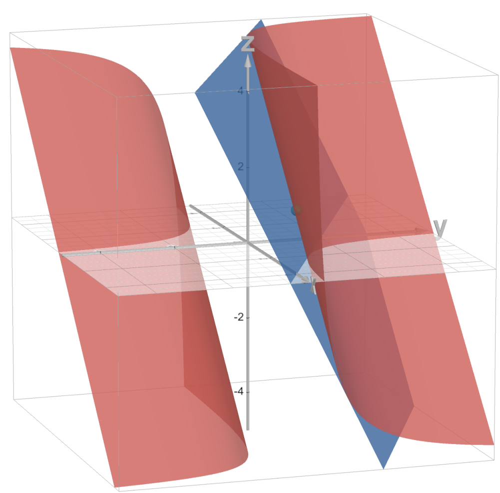
3. Tangent Planes.

(a) Find the equation of the tangent plane to the surface $xy + yz = 2$ at the point $(x,y,z) = (1,1,1)$. [Hint: The normal vector is a gradient vector.]

This surface has the form $f(x,y,z)$ =constant, where $f(x,y,z) = xy + yz$. Note that $\nabla f(x,y,z) = \langle y, x+z, y\rangle$. So the equation of the tangent plane at $(1,1,1)$ is

$$\begin{aligned} \nabla f(1,1,1) \bullet \langle x-1, y-1, z-1\rangle &= 0 \\ \langle 1,2,1\rangle \bullet x-1, y-1, z-1\rangle &= 0 \\ (x-1) + 2(y-1) + (z-1) &= 0 \\ x+2y+z &= 4. \end{aligned}$$

Picture:[1]



(b) Find the equation of the tangent plane to the surface $\mathbf{r}(u,v) = (u,v,uv)$ at the point $\mathbf{r}(2,3) = (2,3,6)$. [Hint: The normal vector has the form $\mathbf{r}_u \times \mathbf{r}_v$.]

To find a normal vector we compute the cross product of two tangent vectors:

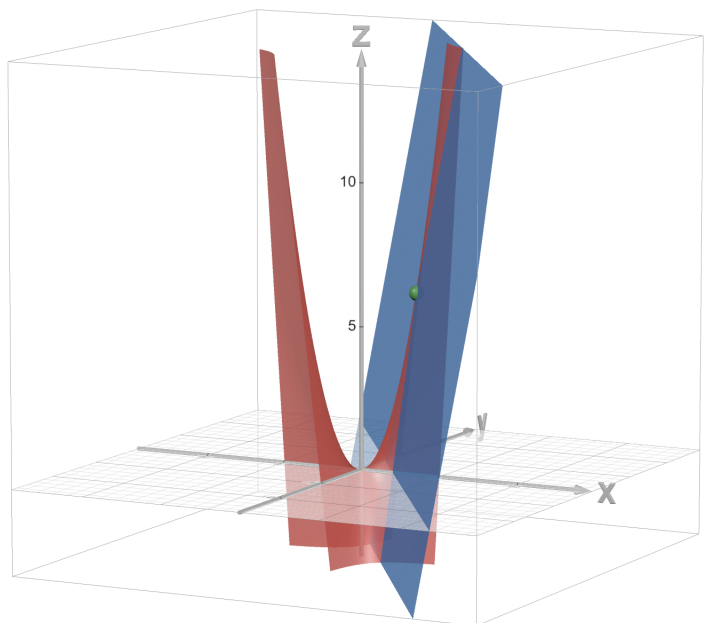
$$\begin{aligned} \mathbf{r}_u &= \langle 1,0,v\rangle, \\ \mathbf{r}_v &= \langle 0,1,u\rangle, \\ \mathbf{r}_u \times \mathbf{r}_v &= \langle -v,-u,1\rangle. \end{aligned}$$

[1] `https://www.desmos.com/3d/w8i7nozgr8`

The tangent vector at the point $(2, 3, 6)$, i.e., when $(u, v) = (2, 3)$ is $\langle -3, -2, 1\rangle$. Hence the equation of the tangent plane at the point $(2, 3, 6)$ is

$$\begin{aligned} \langle -3, -2, 1\rangle \bullet \langle x-2, y-3, z-6\rangle &= 0 \\ 3(x-2) + 2(y-3) - (z-6) &= 0 \\ 3x + 2y - z &= 6. \end{aligned}$$

Picture:[2]



4. Linear Approximation. The base of a rectangular box is a square of side length r and the height is h, so the volume of the box is $V = r^2 h$.

(a) Compute the differential dV in terms of r, h, dr and dh.

We use the multivariable chain rule:

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh \\ &= 2rh dr + r^2 dh. \end{aligned}$$

(b) Suppose we know that $r = h = 1$ cm and that each of r and h has an uncertainty of 0.1 cm. Estimate the uncertainty in the volume V.

Taking $r = h = 1$ and $dr = dh = 0.1$ gives

$$dV = 2(1)(1)(0.1) + (1)^2(0.1) = 0.3 \text{ cm}^3.$$

We can interpret this as the approximate uncertainty in our computation of $V = (1)^2(1) = 1$. We could say that

$$V = 1 \pm 0.3 \text{ cm}^3.$$

[2] https://www.desmos.com/3d/fbxhx6adr3

5. Two Variable Optimization. Find all local maxima, local minima and saddle points for the following functions.

(a) $f(x, y) = xy$

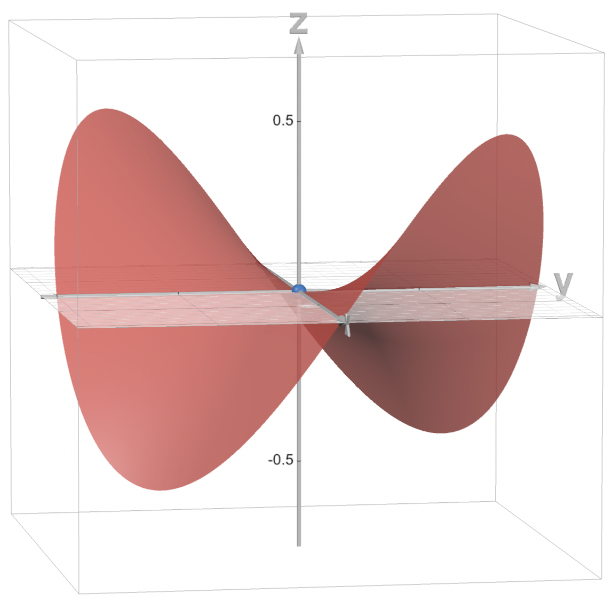
Setting the gradient equal to $\langle 0, 0\rangle$ gives

$$\begin{aligned}\nabla f(x, y) &= \langle 0, 0\rangle \\ \langle y, x\rangle &= \langle 0, 0\rangle,\end{aligned}$$

which implies that the only critical value is $(x, y) = (0, 0)$. To determine the nature of this critical point, we compute the Hessian determinant:

$$\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = -1.$$

Since the determinant is always negative we conclude that $(0, 0)$ is a saddle point. Picture:[3]



(b) $f(x, y) = x^2 + y^2$

Setting the gradient equal to $\langle 0, 0\rangle$ gives

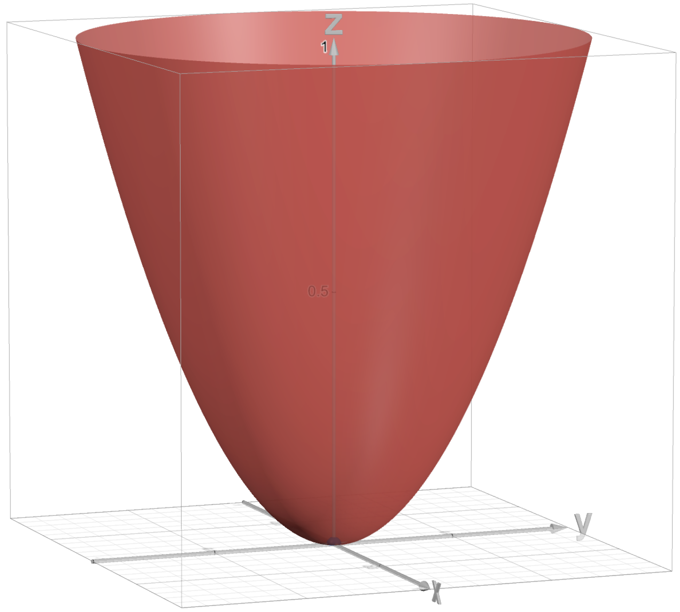
$$\begin{aligned}\nabla f(x, y) &= \langle 0, 0\rangle \\ \langle 2x, 2y\rangle &= \langle 0, 0\rangle,\end{aligned}$$

which implies that the only critical value is $(x, y) = (0, 0)$. To determine the nature of this critical point, we compute the Hessian determinant:

$$\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} = \det\begin{pmatrix} 2 & 0 \\ 0 & 2\end{pmatrix} = 4.$$

Since the determinant is always positive we conclude that $(0, 0)$ is a local max or min. Since $f_{xx}(0, 0) = 2 > 0$, it is a local minimum. Picture:[4]

[3] `https://www.desmos.com/3d/ly5vrdcujc`
[4] `https://www.desmos.com/3d/dj9rkmliif`



6. Integration Using Cartesian Coordinates.

(a) Integrate $f(x, y) = x^2 + y^2$ over the rectangle with $0 \le x \le 2$ and $0 \le y \le 3$.

$$\begin{aligned}\int_0^3 \left(\int_0^2 (x^2 + y^2)\,dx\right)\,dy &= \int_0^3 \left[\frac{1}{3}x^3 + y^2 x\right]_0^2 dy \\ &= \int_0^3 \left(\frac{8}{3} + 2y^2\right) dy \\ &= \left[\frac{8}{3}y + \frac{2}{3}y^3\right]_0^3 \\ &= 8 + 2 \cdot 9 \\ &= 26.\end{aligned}$$

(b) Integrate $f(x, y) = xy$ over the region defined by $0 \le x \le 1$ and $x^2 \le y \le x$.

$$\begin{aligned}\int_0^1 \left(\int_{x^2}^x xy\,dy\right)\,dx &= \int_0^1 \left[\frac{1}{2}xy^2\right]_{x^2}^x dx \\ &= \frac{1}{2}\int_0^1 (x^3 - x^5)\,dx \\ &= \frac{1}{2}\left[\frac{1}{4}x^4 - \frac{1}{6}x^6\right]_0^1 \\ &= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) \\ &= 1/24.\end{aligned}$$

7. Integration Using Polar and Cylindrical Coordinates.

(a) Use polar coordinates to compute the area of the unit disk $x^2 + y^2 \leq 1$.

We can parametrize the disk by $x = r\cos\theta$ and $y = r\sin\theta$ with $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Then since $dxdy = r\,drd\theta$ we have

$$\begin{aligned}
\text{Area} &= \iint 1\,dxdy \\
&= \iint r\,drd\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 r\,dr \\
&= 2\pi \left[\frac{1}{2}r^2\right]_0^1 \\
&= \pi.
\end{aligned}$$

(b) Use cylindrical coordinates to compute the volume of the cone defined by $x^2+y^2 \leq 1$ and $0 \leq z \leq 1 - \sqrt{x^2+y^2}$.

We can parametrize the cone using $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$ with $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1 - r$. Then since $dxdydz = r\,drd\theta dz$ we have

$$\begin{aligned}
\text{Volume} &= \iiint 1\,dxdydz \\
&= \iiint r\,drd\theta dz \\
&= \int_0^{2\pi} d\theta \int_0^1 r\left(\int_0^{1-r} dz\right) dr \\
&= 2\pi \int_0^1 r(1-r)\,dr \\
&= 2\pi \int_0^1 (r - r^2)\,dr \\
&= 2\pi \left[\frac{1}{2}r^2 - \frac{1}{3}r^3\right]_0^1 \\
&= 2\pi \left(\frac{1}{2} - \frac{1}{3}\right) \\
&= \pi/3.
\end{aligned}$$

8. Conservative Vector Fields. Consider the scalar function $f(x,y) = \frac{1}{x+y}$.

(a) Compute the gradient vector field $\nabla f(x,y)$.

Since $f_x = -1/(x+y)^2$ and $f_y = -1/(x+y)^2$ we have

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \left\langle \frac{-1}{(x+y)^2}, \frac{-1}{(x+y)^2} \right\rangle = \frac{-1}{(x+y)^2}\langle 1, 1\rangle.$$

(b) Integrate the vector field $\nabla f(x,y)$ along the path $\mathbf{r}(t) = (0,1)+t(2,3)$ for $0 \le t \le 1$. [Hint: There is a shortcut.]

The Fundamental Theorem of Line Integrals says that the integral of ∇f along any path equals $f(\text{end point}) - f(\text{start point})$. In our case,

$$\int_0^1 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = f(2,4) - f(0,1)$$
$$= \frac{1}{6} - \frac{1}{1} = -5/6.$$

To compute this the hard way, note that $\mathbf{r}(t) = (2t, 1+3t)$ and $\mathbf{r}'(t) = (2,3)$, hence

$$\begin{aligned}
\int_0^1 \nabla f(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int_0^1 -\frac{-1}{((2t)+(1+3t))^2}\langle 1,1\rangle \bullet \langle 2,3\rangle\,dt \\
&= \int_0^1 \frac{-5}{(5t+1)^2}\,dt \\
&= \int_1^6 \frac{-1}{u^2}\,du && u = 5t+1, du = 5dt \\
&= \left[\frac{1}{u}\right]_1^6 \\
&= \frac{1}{6} - \frac{1}{1} = -5/6.
\end{aligned}$$

9. Green's Theorem. Consider the vector field $\mathbf{F}(x,y) = \langle P, Q\rangle = \langle x^2+y^2, xy\rangle$.

(a) Compute the curl $Q_x - P_y$.

The curl is $Q_x - P_y = (xy)_x - (x^2+y^2)_y = y - 2y = -y$.

(b) Integrate $Q_x - P_y$ over the half disk defined by $x^2+y^2 \le 1$ and $0 \le y$. [Hint: Use polar coordinates.]

We can parametrize the half disk by $x = r\cos\theta$ and $y = r\sin\theta$ with $0 \le r \le 1$ and $0 \le \theta \le \pi$. Since $dxdy = r\,drd\theta$ we have

$$\begin{aligned}
\iint (Q_x - P_y)\,dydx &= \iint -y\,dydx \\
&= \iint -r\sin\theta\, r\,drd\theta \\
&= -\int_0^1 r^2\,dr \int_0^\pi \sin\theta\,d\theta \\
&= -\left[\frac{1}{3}r^3\right]_0^1 [-\cos\theta]_0^\pi \\
&= -\frac{1}{3}[-\cos(\pi)+\cos(0)] \\
&= -\frac{1}{3}[-(-1)+(1)] \\
&= -2/3.
\end{aligned}$$

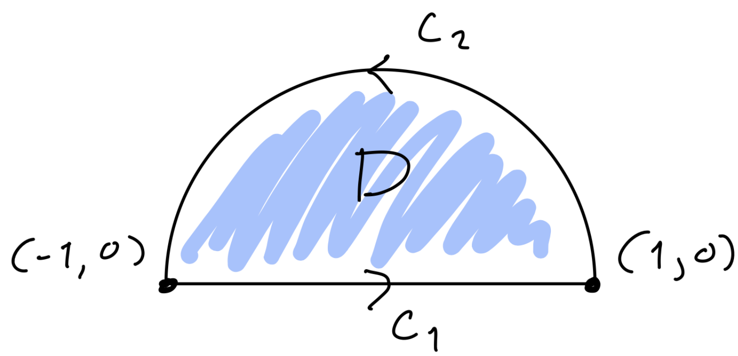
(c) Compute the integral of $\mathbf{F}(x,y) = \langle x^2+y^2, xy\rangle$ along the path $\mathbf{r}(t) = (-1,0)+t(2,0)$ for $0 \le t \le 1$.

Since $\mathbf{r}(t) = (2t-1, 0)$ and $\mathbf{r}'(t) = (2,0)$ we have

$$\begin{aligned}
\int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\,dt &= \int_0^1 \langle (2t-1)^2, 0\rangle \bullet \langle 2,0\rangle\,dt \\
&= 2\int_0^1 (2t+1)^2\,dt \\
&= 2\int_0^1 (4t^2-4t+1)\,dt \\
&= 2\left[\frac{4}{3}t^3 - 2t^2 + t\right]_0^1 \\
&= 2\left(\frac{4}{3} - 2 + 1\right) \\
&= 2/3.
\end{aligned}$$

(d) Compute the integral of $\mathbf{F}(x,y) = \langle x^2+y^2, xy\rangle$ along the path $\mathbf{r}(t) = (\cos t, \sin t)$ for $0 \le t \le \pi$. [Hint: There might be a shortcut.]

If D is the half disk from (b) and if C_1, C_2 are the oriented paths from (c),(d), respectively, then we have $\partial D = C_1 + C_2$. Picture:



Hence Green's Theorem gives

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \bullet \mathbf{T} &= \iint_D (Q_x - P_y)\,dxdy \\
\int_{C_1} \mathbf{F} \bullet \mathbf{T} + \int_{C_2} \mathbf{F} \bullet \mathbf{T} &= \iint_D (Q_x - P_y)\,dxdy \\
2/3 + \int_{C_2} \mathbf{F} \bullet \mathbf{T} &= -2/3 \\
\int_{C_2} \mathbf{F} &= -4/3.
\end{aligned}$$

Remark: The integral can also be computed directly, but you need to know the antiderivative of $\sin^3 t$, or something similar.