Analysis Preliminary Exam

May 2013

Show your work.

Do 10 out of the following 11 problems

— Use a new page for each problem —

P 1.

State and prove the following theorems:

(i) Fermat's theorem on the extreme value points of a differentiable function

(ii) Rolle's theorem. You may assume the Extreme Value Theorem without proof.

P 2.

State without proof the Implicit Function Theorem.

P 3.

Let (x_n), $n \geq 0$ be a bounded sequence of real numbers. For each n, define $y_n = \sup_{k \geq n} x_k$.

(i) Show that (y_n) has a limit $y \in \mathbb{R}$.

(ii) Show that $\forall \epsilon > 0$, $\exists\, N_\epsilon$ such that $x_n < y + \epsilon$ if $n \geq N_\epsilon$.

(iii) If $\lim_{n\to\infty} x_n = x$, then $x = y$.

P 4.

Let $f : \mathbb{R} \to \mathbb{R}$ have two derivatives and suppose that $f(x) \leq 0$ and $f''(x) \geq 0$ for all x. Prove that f is constant.

P 5.

Let (a_n) be a sequence of positive numbers such that $\sum a_n$ diverges. Show that

(i) $\sum \frac{a_n}{1+n^2 a_n}$ converges,

(ii) $\sum \frac{a_n}{1+a_n}$ diverges.

P 6.

Let $f_n(x) = \frac{2nx}{1+n^2 x}$, $x \in [0,1]$ and $n \geq 1$.

(i) Determine the pointwise limit $f(x)$ of the sequence.

(ii) Is the limit uniform?

P 7.

Let

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(\sqrt{n}x)}{n^2} \,.$$

Show that f is continuous on $\mathbb{R}$ and the series can be differentiated term by term.

P 8.

Show that if $f : [0, 1] \to \mathbb{R}$ is continuous and non-negative and its maximum is M, then

(i) for any $\epsilon > 0$, there exists a closed interval $[a, b] \subseteq [0, 1]$ such that $M - \epsilon < f(x)$, $x \in [a, b]$.

(ii)

$$\lim_{n\to\infty} \Big(\int_0^1 f^n(x)dx\Big)^{\frac{1}{n}} = M \,.$$

P 9.

Let D and D' be subsets of $\mathbb{R}^d$.

(i) Show that if $\Phi : D \to D'$ is continuous with continuous inverse and D is pathwise connected then D' is pathwise connected.

(ii) Show that the ellipsoid $2x^2+y^2+5z^2 = 10$ is pathwise connected.

P 10.

Find the maximum and minimum of $f(x, y, z) = x^2 + y^2 + z^2$ when $2x^2 + y^2 + 3z^2 \leq 1$.

P 11.

Let f be given by $f(x_1, x_2) = (y_1, y_2)$ with

$$y_1 = \frac{x_1}{1 + x_1 + x_2} \,, \quad y_2 = \frac{x_2}{1 + x_1 + x_2} \,,$$

ond the set $D = \{(x_1, x_2)|x_1 + x_2 > -1\} \subseteq \mathbb{R}^2$.

Show that the inverse function theorem is applicable at any point of D and calculate the inverse explicitly.