IMMORTAL PARTICLE FOR A CATALYTIC BRANCHING PROCESS†

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Abstract. We study the existence and some asymptotic properties of a conservative branching particle system for which birth and death are triggered by contact with a set. Sufficient conditions for the process to be non-explosive are given, solving a long standing open problem. With probability one, it is shown that only one ancestry survives. In special cases, the evolution of the surviving particle is studied and for a two particle system on a half line we derive explicitly the transition function of a chain representing the position at successive branching times.

1. **Introduction**

This paper is the second part of an effort to characterize the non-explosiveness and ergodic properties of a class of stochastic processes built by piecing together countably many consecutive episodes of a driving process killed upon contact with a set (catalyst), which is restarted at a random point of the state space to be prescribed according to the particular evolution model by a redistribution probability measure. The first part [11] looks at a number of models that need a finite number of jumps before entering a certain center of the state space (a small set in the sense of Doeblin theory). This paper is dedicated to the harder example of the N particle system with Fleming-Viot dynamics introduced in [3] for Brownian motions. Similarly to the Wright-Fisher model, a killed particle is replaced by having one of the surviving particles branch; this can be interpreted as a jump to the location of one of the survivors, chosen uniformly. We admit general diffusions with

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smooth coefficients as a driving process, and in Theorem 1 the domain of evolution may be unbounded. Lemma 1 is the main tool of the proof. In the context of the Fleming-Viot it shows that the process enters almost surely a configuration with *at least one particle* in the center of the set. The proof does not require the uniform distribution at jump time, a lower bound on the redistribution probabilities being sufficient (Definition 1). Perturbations of the redistribution probabilities appear naturally in large deviations estimates from the hydrodynamical limit [9].

Our three main results are Theorem 1, which proves that for sufficiently large N, the system is non-explosive on domains with regularity prescribed in Definition 2; Theorem 2 which proves the geometric ergodicity using a comparison with a process without jumps obtained by coupling; and Theorem 3, establishing the existence of a unique infinite continuous path, or ancestry line - the immortal particle in the sense of [5, 6]. For a discussion and recent developments the reader is referred to [2] and the references therein. Our interest in the model was based on the scaling properties of the branching model [10]. The hydrodynamic limit (law of large numbers for the empirical measures as objects on the Skorohod space) has been explored in [7, 1] as a tool to study the quasi-invariant measures of a killed process, providing an important application of the Fleming-Viot mean-field redistribution dynamics.

Let D be an open connected set in $\mathbb{R}^d$ with regular boundary ∂D and $((\tilde{x}(t))_{t\geq 0}$ a diffusion on D absorbed at the boundary, generated by the second order strictly elliptic operator $\mathcal{L}$. We shall assume that the diffusion coefficients are smooth up to the boundary, i.e. belong to the $C^\infty(\bar{D})$. Naturally, lower regularity may be considered but the problems considered are difficult enough for the Laplacian. This setup can accommodate with minor changes the case of a diffusion with some boundary conditions (i.e. reflecting) on a subset of its topological boundary. In that case ∂D will denote without loss of generality, the *absorbing boundary*, where the process is killed upon arrival. Under these assumptions $P_x(\tau^D > 0) = 1$ for all $x \in D$, where $\tau^D = \inf\{t > 0 \,|\, x(t) \in D^c\}$ is the hitting time of D^c, the complement of D, and the transition probabilities $P^D(t, x, dy)$ will have a density

$$P_x(\tilde{x}(t) \in dy\,,\, \tau^D > t) = P^D(t, x, dy) = p^D(t, x, y)dy\,. \tag{1.1}$$

We note that the harmonic measures $P_x(x(\tau^D-) \in d\xi)$ are absolutely continuous with respect to the Lebesgue measure on the boundary $\lambda_0(d\xi)$, $\xi \in \partial D$.

In addition, for any $\xi \in \partial D$ we have a probability measure $\nu(\xi, dx)$ on D such that $\xi \to \nu(\xi, dx)$ is measurable with respect to the Borel σ-algebras of ∂D and of $M_1(D)$, where $M_1(D)$ denotes the space of probability measures on D with the topology of convergence in distribution.

Constructively, we define a Markov process $x(t)$, $t \geq 0$, starting at $x \in D$, as follows. We set $x_0 := x$ and $\tau_0 := 0$. The process follows the diffusion P^D starting at x_0 up to $\tau_1 := \tau_1^D$, which means $x(t) := \tilde{x}(t)$ for $0 \leq t < \tau_1$. As soon as it reaches ∂D at $\xi_0 = x(\tau_1-)$ it instantaneously jumps to a random point $x_1 \in D$, independent of the process $x(t)$, with distribution $\nu(\xi_0, dx)$. We continue the motion according to the diffusion p^D starting at x_1 until $\tau_2 = \inf\{t > \tau_1 \,|\, x(t) \in D^c\}$. We set $x(t) = \tilde{x}(t - \tau_1)$ on $\tau_1 \leq t < \tau_2$, where evidently $\tau_2 - \tau_1 = \tau_2^D$ and so on. Since $P_x(\tau^D > 0) = 1$ for all $x \in D$ we have that τ_l is strictly increasing in $l \geq 0$. It is possible that $\tau_{l'} = \infty$ for a given l', in which case $\tau_l \equiv \infty$ for all $l \geq l'$. Without loss of generality, let $l' = \inf\{l \geq 1 | \tau_l = \infty\}$ and we denote l^* the total number of jumps; obviously $l^* = l' - 1$. We denote $\tau^* = \lim_{l \to \infty} \tau_l \leq \infty$.

In the following, for a sufficiently small $\delta > 0$, we denote $D_\delta = \{x \in D \,|\, d(x, \partial D) > \delta\}$. The underlying diffusion will be assumed to satisfy the uniform bound on the exit time from a vicinity of the boundary D_δ^c, trivial for a bounded D,

$$\lim_{t \to \infty} \sup_{x \in \bar{D}_\delta^c} P_x(\tau^{D_\delta^c} > t) = 0 \,. \tag{1.2}$$

We are interested in conditions guaranteeing that $x(t)$ is *non-explosive* or, equivalently, does not finish in finite time with positive probability (1.3)

$$\forall x \in D \,, \qquad P_x(\lim_{l \to \infty} \tau_l^D = \infty) = 1 \,. \tag{1.3}$$

Lemma 1 contains the key element in the proof of non-explosiveness exhibited by the function $\ln \Phi(x)$, where $\Phi(x)$ is intuitively emulating the distance to a subset A of ∂D, with properties $\Phi(x) > 0$ in D and $\Phi(x) = 0$ on A. The reader may want to think of D' as a subset of $D \setminus \bar{D}_\delta$ representing the "worst case scenario" for survival because $\partial D' \supseteq A$, in other words a set where the process may have the highest chance of extinction.

Besides technical assumptions contained in (i), properties (ii) and (iii) guarantee that $\ln \Phi(x(t))$ is a (local) sub-martingale. More importantly, $\ln \Phi(x(t))$ experiences a strictly positive jump (iii) on the boundary, implying that the process pays a "price" for each jump.

For a set $D_1 \subseteq D$, we define $\alpha(D_1) = \inf\{t > 0 \,|\, x(t) \in \bar{D}_1\}$, with $\alpha(D_1) = +\infty$ if $x(t)$ never hits $\bar{D}_1$. Let $l(D_1) = \max\{l \,|\, \tau_l \leq \alpha(D_1)\}$, the number of jumps until the process hits $\bar{D}_1$.

Lemma 1. *Assume there exists a (possibly empty) closed subset A of the boundary ∂D with $\lambda_0(A) = 0$, an open subset $D' \subseteq D$ and a bounded real function $\Phi \in C^2(D') \cap C(\bar{D})$ with the properties (i) $\Phi(x) > 0$ on $\bar{D} \setminus A$ and $\Phi(x) = 0$ on A; (ii) $\mathcal{L} \ln \Phi(x) \geq 0$ for all $x \in D'$ and (iii) $U = \inf_{\xi \in (\partial D \cap \partial D') \setminus A} U(\xi) > 0$, where*

$$U(\xi) = \int_D \ln \Phi(x) \nu(\xi, dx) - \ln \Phi(\xi)\,. \tag{1.4}$$

Then $E_x[l(D \setminus D')] \leq U^{-1}[\sup_{x' \in \bar{D}}\{\ln \Phi(x')\} - \ln \Phi(x)] < \infty$ for all $x \in D'$. If, in addition, there exists $\delta > 0$ such that $D' \subseteq D \setminus D_\delta$, then $P_x(\alpha(D \setminus D') < \infty) = 1$ for all $x \in D'$.

Proof. Step 1. We show that $\ln \Phi(x(t \wedge \alpha(D \setminus D')))$, $t \geq 0$, $x(0) = x \in D'$ is a local $(\mathcal{F}_t)$ sub-martingale. Condition (ii) shows that $\ln \Phi(x(t))$ is a sub-martingale as long as $x(t) \in G'$ and (iii) shows that it is a sub-martingale at the boundary of D' shared with ∂D. This proves the statement up to the first hitting time of $D \setminus D'$. Since $\Phi(x) = 0$ on A, we create a localizing sequence on $\bar{D} \setminus A$. Due to $\lambda_0(A) = 0$, there exists a nested sequence of open sets $B_k \subseteq \mathbb{R}^d$, $B_k \supseteq A$, such that for all $k \geq 0$, $d(y, A) < 1/k$ when $y \in B_k$. We may assume without loss of generality that $x \notin B_0$ and $B_0 \subseteq D \setminus D_\delta$. We claim that if $\tau(B_k^c) = \inf\{t > 0 \,|\, x(t) \in \bar{B}_k\}$ and (with a slight abuse of notation) $\tau(B_\infty^c) = \lim_{k\to\infty} \tau(B_k^c)$, then $P_x(\tau(B_\infty^c) \geq \tau^*) = 1$ for all $x \in D \setminus B_0$. Assume $\tau(B_\infty^c) < \tau^*$. The sequence is $\tau(B_k^c)$ is non-decreasing, but we want to show that it cannot be constant from a certain rank on. If this would be the case, $\tau(B_k^c) = \tau(B_{k_0}^c)$ for all $k \geq k_0$ and there exists l such that $\tau(B_{k_0}^c) \in [\tau_{l-1}, \tau_l)$. Consequently $x(\tau(B_{k_0}^c)) \in D$ yet $d(x(\tau(B_{k_0}^c)), A) \leq 1/k$ for all $k \geq k_0$, thus $x(\tau(B_{k_0}^c)) \in A$, a contradiction. Without loss of generality, we assume that the sequence $\tau(B_k^c)$ is *strictly increasing.* There are two possibilities: Either $(\tau(B_k^c))$, $k \geq 0$ has only finitely many points in each episode $[\tau_{l-1}, \tau_l)$, $l \geq 1$, or there exists $l_A < \infty$ with infinitely many $\tau(B_k^c)$ in $[\tau_{l_A-1}, \tau_{l_A})$. In the first case $\tau(B_\infty^c) \geq \tau^*$, and we are done. In the second case, $\tau(B_\infty^c) \neq \tau_{l_A-1}$, so there are two scenarios: Either $\tau(B_\infty^c) \in (\tau_{l_A-1}, \tau_{l_A})$, or $\tau(B_\infty^c) = \tau_{l_A}$. In both, the process $x(t)$ has continuous paths on $(\tau_{l_A-1}, \tau_{l_A})$ and $d(x(\tau(B_k^c)), A) \leq 1/k$ for an infinite subsequence, which implies that the path of the diffusion killed at the boundary has a limit point on A. This event has zero probability on any episode and there are countably many episodes. By choosing the localizing sequence $\tau(B_k^c) \wedge \alpha(D \setminus D')$, $k \geq 0$ we proved *Step 1.*

Step 2. Fix $x(0) = x \in D'$. Denote $M(\Phi) = \sup_{x' \in \bar{D}}\{\ln \Phi(x')\}$, let m be a positive integer, $T > 0$ and put $\tau'_j = \tau_{j\wedge m} \wedge (\tau(B_k^c) \wedge \alpha(D \setminus D')) \wedge T$ for all $j \geq 0$ and $\tau(B_k^c)$, k fixed at the moment, as in *Step 1.* With this notation, the summations below are finite, and we can write

$$M(\Phi) - \ln \Phi(x) \geq E_x[\ln \Phi(x(\tau_{l(D')\wedge m} \wedge (\tau(B_k^c) \wedge \alpha(D \setminus D')) \wedge T)) - \ln \Phi(x(0))] \tag{1.5}$$

$$= E_x[\sum_{j=1}^{l(D\setminus D')} \ln \Phi(x(\tau'_j)) - \ln \Phi(x(\tau'_{j-1}))] \tag{1.6}$$

$$= \sum_{j=1}^{l(D\setminus D')} E_x[\ln \Phi(x(\tau'_j)) - \ln \Phi(x(\tau'_j -))] + \sum_{j=1}^{l(D\setminus D')} E_x[\ln \Phi(x(\tau'_j -)) - \ln \Phi(x(\tau'_{j-1}))] \tag{1.7}$$

The second term of (1.7) representing the diffusive time interval $[\tau'_{j-1}, \tau'_j -)$ is nonnegative by the sub-martingale property. The first term, representing the jump at τ'_j is bounded below by

$$E_x[\sum_{j=1}^{m} E_x[\ln \Phi(x(\tau'_j)) - \ln \Phi(x(\tau'_j -))|\mathcal{F}_{\tau'_j -}]] \tag{1.8}$$

$$= \sum_{j=1}^{m} E_x[E_{\mathbf{x}(\tau'_j -)}[\ln \Phi(x(\tau'_j)) - \ln \Phi(x(\tau'_j -))]] \,, \tag{1.9}$$

where we used the strong Markov property. Due to the choice of the times τ'_j, the sequence τ'_j becomes constant for $j \geq m$ (or possibly earlier on). Let $\eta(s)$, $s > 0$ be equal to one if s is an actual jump time of the process $x(s) - x(s-) \neq 0$ and to zero if it is a continuity point. With $J(t)$ denoting the number of jumps up to time t,

$$E_{x(\tau'_j -)}[\ln \Phi(x(\tau'_j)) - \ln \Phi(x(\tau'_j -))] \geq U\eta(\tau'_j) \tag{1.10}$$

leading to the lower bound $UE_x[J((\tau(B_k^c) \wedge \alpha(D \setminus D')) \wedge T) \wedge m]$ for line (1.9). We have shown

$$E_x[J((\tau(B_k^c) \wedge \alpha(D \setminus D')) \wedge T) \wedge m] \leq U^{-1}(M(\Phi) - \ln \Phi(x)) \,, \tag{1.11}$$

with right hand side not depending on T, k, and m. We let $m \to \infty$, then $T \to \infty$ and finally $k \to \infty$ to obtain $E[J(\tau^* \wedge \alpha(D \setminus D'))] \leq U^{-1}(M(\Phi) - \ln \Phi(x))$. Set $H = \{\alpha(D \setminus D') \geq \tau^*\}$. Of course, this can happen only if $\alpha(D \setminus D') = \infty$, by construction. Nonetheless, this still allows the possibility that $\tau^* < \infty$. But we have $H \subseteq \{J(\tau^*) < \infty\}$, a set of measure zero, which shows that $\alpha(D \setminus D') < \tau^*$ and thus

$$E_x[J(\alpha(D \setminus D'))] = E_x[l(D \setminus D')] \le U^{-1}[M(\Phi) - \ln \Phi(x)] < \infty\,. \tag{1.12}$$

Step 3. If $x \in D'$,

$$P_x(\alpha(D \setminus D') > T) \le P_x(\alpha(D \setminus D') > T\,,\, l(D \setminus D') \le l) + P_x(l(D \setminus D') > l) \tag{1.13}$$

$$\le l \sup_{x' \in D_\delta^c} P_{x'}(\tau^{D_\delta^c} > \frac{T}{l}) + \frac{1}{Ul}(M(\Phi) - \ln \Phi(x))\,, \tag{1.14}$$

where we used (1.2). For l fixed, we let $T \to \infty$ and obtain $\limsup_{T\to\infty} P_x(\alpha(D \setminus D') > T) \le \frac{1}{Ul}(M(\Phi) - \ln \Phi(x))$. Let $l \to \infty$ to prove the second claim.

□

2. **The Fleming-Viot redistribution case**

In this setup, $N \ge 2$ is a positive integer, the domain $D = G^N$, with G a region in $\mathbb{R}^q$, $d = Nq$ with *regular boundary* ∂G. The process $\{\mathbf{x}(t)\}_{t\ge 0}$ has components $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$ (called particles), each $\{x_i(t)\}_{t\ge 0}$, $1 \le i \le N$ evolving in G as a q - dimensional diffusion with jumps at the boundary ∂G to be described in the following. As before, the process $\{\mathbf{x}(t)\}_{t\ge 0}$ is adapted to a right-continuous filtration $\{\mathcal{F}_t\}_{t\ge 0}$. For $\xi \in \partial D$ we write $I(\xi) = \{i \,|\, \xi_i \in \partial G\}$ and $\xi^{ij} \in G^N$ denotes the vector with the same components as ξ with the exception of ξ_i which is replaced by ξ_j.

When a particle x_i reaches ∂G at τ, it jumps instantaneously to the location of one of the remaining particles x_j, $1 \le j \le N$, $j \ne i$ (there are no simultaneous boundary visits a.s.) with probabilities $p(\mathbf{x}(\tau-), j)$, $1 \le j \le N$, having only the restriction $p(\mathbf{x}(\tau-), i) = 0$. It is obviously possible to allow positive probabilities for stopping at the boundary, a standard construction being to allow an exponential time before attempting a new jump. However we do not pursue this approach here since it rather obscures the natural question of non-explosiveness. There is no real ambiguity concerning points on the "edges" of the boundary (i.e. when at least two components are on ∂G, or $|I(\xi)| \ge 2$) since the underlying diffusion does not visit a.s. sets of co-dimension greater than two as soon as it starts at points $\mathbf{x} \in D$. We shall not start the process on the boundary. However we shall define $\nu_\xi(d\mathbf{x})$ for all $\xi \in \partial D$ in (2.1), even though the definition on edges may be arbitrary. More precisely, there exist measurable functions $\partial G^N \ni \xi \to p_{ij}(\xi) \in [0,1]$, indexed by $1 \le i,j \le N$ such that $p_{ij}(\xi) = 0$ whenever $i = j$ and $\sum_j p_{ij}(\xi) = 1$ such that

$$\forall \xi \in \partial G^N\,, \qquad \nu(\xi, d\mathbf{x}) = \frac{1}{|I(\xi)|} \sum_{i \in I(\xi)} \sum_{j=1}^{N} p_{ij}(\xi) \delta_{\xi^{ij}}(d\mathbf{x})\,. \tag{2.1}$$

Definition 1. *We shall say that the redistribution probabilities $p_{ij}(\xi)$ are* non-degenerate *if they are bounded away from zero uniformly; i.e. there exists $p_0 > 0$ independent of $\xi \in \partial G^N$, such that $p_{ij}(\xi) \ge p_0$, $1 \le i,j \le N$, $i \ne j$.*

Remark. 1) Except on the edges of $D = G^N$, formula (2.1) does not have a proper average over $i \in I(\xi)$. The definition is consistent over all $\xi \in \partial D$.

2) The most natural choice of $p_{ij}(\xi)$ is uniform $p_{ij}(\xi) = (N-1)^{-1}$, $j \ne i$, $\xi \in \partial G$. In that case $p_0 = (N-1)^{-1}$.

3) The definition (2.1) is *not necessarily continuous* as a function in ξ into $M_1(\bar{D})$ with the topology of weak convergence of measures; the reader may check the case $N = 3$, $d = 1$ with the redistribution measures from 2).

4) Assume D is bounded. Then $\bar{D}$ is compact, and the family of measures $(\nu_\xi(d\mathbf{x}))_{\xi \in \partial D}$ is tight. Nonetheless, limit points might be concentrated on ∂D, which raise the danger that the process is explosive.

5) Definition 1 can be relaxed, with proper care for the regularity of the domain, as follows. It is only the $p_{ij}(\xi)$ corresponding to the j with *maximum distance* from the boundary that needs a lower bound.

We shall further assume that the particles $x_i(t)$ evolve *independently* between jumps, each following a diffusion with generator L on $\mathbb{R}^q$ killed at the boundary ∂G. More specifically

$$Lu(x) = \sum_{1 \le \alpha \le q} b^\alpha(x) \frac{\partial u}{\partial x^\alpha}(x) + \frac{1}{2} \sum_{1 \le \alpha,\beta \le q} a^{\alpha,\beta}(x) \frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}(x)\,, \quad u \in C_0(\mathbb{R}^q) \cap C^2(\mathbb{R}^q)\,, \tag{2.2}$$

with coefficients $\{b^\alpha(x)\}_\alpha$, $\{a^{\alpha,\beta}(x)\}_{\alpha,\beta}$ in $C^\infty(\mathbb{R}^q)$. With the notation $\sigma(x)\sigma^*(x) = a(x)$ (the star stands for the matrix transposition), the coefficients are uniformly bounded, with L strictly elliptic

$$|b^\alpha(x)| \le ||b||\,, \qquad 0 < \sigma_0^2 ||v||^2 \le ||\langle \sigma(x)\sigma^*(x)v, v\rangle|| \le ||\sigma||^2 ||v||^2\,, \quad v \in \mathbb{R}^q\,, \tag{2.3}$$

where $||b||$, σ_0, $||\sigma||$ do not depend on x, α, β. Under these conditions, there exists a family of Brownian motions $\{w_i^\beta(t)\}_{1 \le \beta \le q}$, mutually independent in i as well as β, adapted to

$(\mathcal{F}_t)$, such that between successive jumps, the N components $x_i(t) = (x_i^1(t), \ldots, x_i^q(t)) \in G$, where $(x_i^\alpha(t))_{1\le\alpha\le q}$ are solutions to the stochastic differential equations

(2.4)
$$dx_i^\alpha(t) = b^\alpha(x_i(t))dt + \sum_{1\le\beta\le q} \sigma^{\alpha,\beta}(x_i(t))dw_i^\beta(t)\,, \quad 1 \le \alpha, \beta \le q\,, \qquad x_i(0) = x_{i0} \in G\,,$$

for all $1 \le i \le N$.

2.1. **Domain regularity.** Until this point we only required that ∂G be regular, guaranteed, for example, by the *exterior cone condition.* For any regular domain G and any one-particle diffusion with smooth coefficients (2.2), if U, V are two open subsets of G with $U \subseteq V \subseteq G$, $T > 0$, we denote by $p_\pm(T, U, V)$ the supremum, respectively infimum over $x \in U$ of $P_x(\tau^V > T)$, where τ^V denotes the first hitting time of ∂V. We start with the following remark. If $U \subset\subset V$ such that $0 < d_- \le d(\partial U, \partial V) \le d_+ < \infty$, then there exist constants $p_\pm(T, U, V)$ such that for all $x \in \bar{U}$

$$0 < p_-(T, U, V) \le P_x(\tau^V > T) \le p_+(T, U, V) < 1\,. \tag{2.5}$$

To check (2.5), we set $w(T, x) = P_x(\tau^V > T)$ on $x \in V$ and note that $(\partial_T - L)w(T, x) = 0$, $0 \le w(T, x) \le 1$ and $w(T, x) = 0$ on ∂V. The lower bound is guaranteed by the maximum principle applied to $w(T, x)$ and the upper bound by applying it to $1 - w(T, x)$.

Definition 2. *We shall say that* G *has a* uniform distance from the boundary *if there exists an open* $G' \subset G$ *such that* $d(\partial G, G \setminus G') > 0$ *and there exists a function* ϕ *such that (i)* $\phi \in C^2(G') \cap C(\bar{G})$ *and all derivatives up to order two are uniformly bounded on* G'*; (ii)* $\phi(x) > 0$, $x \in G'$*; (iii)* $\phi(x) = 0$, $x \in \partial G$*; (iv) there exists a positive constant* c_- *depending on* G' *and* ϕ *only, such that* $||\nabla\phi(x)|| \ge c_-$ *uniformly over* G'*.*

We shall denote the other uniform bounds: There exist positive real constants C_+, c_+ depending on G' and ϕ only, and c_L that may depend, in addition, on the generator L, such that $\phi(x) \le C_+$, $x \in \bar{G}$ and when $x \in G'$, we have both $||\nabla\phi(x)|| \le c_+$ and $|L\phi(x)| \le c_L$.

Theorem 1. *Assume that* G *satisfies the conditions of Definition 2 and the relocation probabilities satisfy the condition in Definition 1. Then, for* N *sufficiently large, the process is non-explosive in the sense of (1.3). More precisely* $N > 2(\frac{||\sigma||c_+}{\sigma_0 c_-})^2$ *and equality may be achieved if* ϕ *is sub-harmonic.*

Proof. The plan is to prove the theorem in two steps. *Step 1* will apply Lemma 1 to $D = G^N$ with $D' = (G')^N$, where $G' = G \setminus \bar{G}_{2^N\delta}$ for some suitably small but fixed $\delta > 0$ and the set $A = \{\xi \in \partial G^N \mid I(\xi) = N\}$ will be the *vertices* of the domain, i.e. the part of the boundary ∂G^N with all components in ∂G. *Step 1* will conclude that the process $\mathbf{x}(t)$ exits in finite time D', with probability one. In *Step 2* we show that once in $D \setminus D'$, the process will hit the set $(\bar{G}_\delta)^N$ in *a finite number of jumps* with probability one. From that point on we apply Lemma 3 from [11] and we are done.

Step 1. Let $(\mathbf{y}(t))$ be the process with one-dimensional components $y_i(t) := \phi(x_i(t))$, $t \geq 0$. We are interested in the logarithm of the radial process $(r(t))$

$$r(t) = \Phi(x(t)), \qquad \Phi(\mathbf{x}) = (\sum_{i=1}^{N} \phi^2(x_i))^{\frac{1}{2}}. \tag{2.6}$$

Using Ito's lemma, the N - dimensional process $(\mathbf{y}(t))$ satisfies the stochastic differential equations

$$dy_i(t) = \tilde{b}_i(t)dt + \tilde{\sigma}_i(t)d\tilde{w}_i(t), \qquad y_i(0) = \phi(x_{i0}), \tag{2.7}$$

where $\{\tilde{w}_i(t)\}_{1\leq i\leq N}$ are Brownian motions adapted to $(\mathcal{F}_t)$ obtained from (2.4) by the representation theorem for continuous martingales. Concretely, $\tilde{b}(t) = (\tilde{b}_i(t))_{1\leq i\leq N}$, $(\tilde{\sigma}_i(t))_{1\leq i\leq N}$ have components

$$\tilde{b}_i(t) = L\phi(x_i(t)), \qquad \tilde{\sigma}_i(t) = ||\sigma^*(x_i(t))\nabla\phi(x_i(t))|| \tag{2.8}$$

with the inequalities

$$0 < \sigma_0^2||\nabla\phi(x_i(t))||^2 \leq \tilde{\sigma}_i^2(t) \leq ||\sigma||^2||\nabla\phi(x_i(t))||^2 \tag{2.9}$$

due to (2.3). By construction, $\Phi(\mathbf{x}) = 0$ if and only if all $\phi(x_i) = 0$. In D', this means only on A. The only conditions on Φ from Lemma 1 that have to be verified are (ii) and (iii).

Between jumps $r(t)$ satisfies

$$dr(t) = B(t)dt + S(t)dW(t), \quad r(0) = ||\phi(x(0))||, \tag{2.10}$$

where $(W(t))$ is a one - dimensional Brownian motion adapted to $(\mathcal{F}_t)$, while the drift $B(t)$ and variance matrix $S(t)$ are given by (here $Tr(A)$ is the trace of the $N \times N$ matrix A)

$$B(t) = \frac{1}{2r(t)}\Big(2\langle\mathbf{y}(t), \tilde{b}(t)\rangle + Tr(\tilde{\sigma}(t)\tilde{\sigma}^*(t)) - \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^2(t)}\Big) \tag{2.11}$$

$$(2.12) \qquad S(t) = \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||}{r(t)}\,.$$

In the formula above $\tilde{\sigma}^*(t)$ is the $N \times N$ diagonal matric with entries $\tilde{\sigma}_i(t)$ from (2.8).

Relations (2.11)-(2.12), the choice of $\ln r$ and Ito's lemma imply that between jumps, and away from the origin (the rigorous argument is given below) $\{\ln(r(t))\}_{t\geq 0}$ is a sub-martingale with respect to $(\mathcal{F}_t)$ as soon as $2r(t)B(t) - S^2(t) \geq 0$. This is equivalent to

$$(2.13) \qquad 2\langle \mathbf{y}(t), \tilde{b}(t)\rangle + Tr(\tilde{\sigma}(t)\tilde{\sigma}^*(t)) - 2\frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^2(t)} \geq 0\,.$$

We note an extra factor of two in front of the last term as opposed to (2.11). In detail,

$$(2.14) \qquad 2\sum_{i=1}^{N} \phi(x_i(t))L\phi(x_i(t)) + \sum_{i=1}^{N} \tilde{\sigma}_i^2(t) - 2\frac{\sum_{i=1}^{N} \tilde{\sigma}_i^2(t)y_i^2(t)}{\sum_{i=1}^{N} y_i^2(t)}$$

has lower bound

$$(2.15) \qquad N\Big(- 2||\phi||c_L + \sigma_0^2(\inf||\nabla\phi(x)||)^2\Big) - 2||\sigma||^2(\sup||\nabla\phi(x)||)^2\,,$$

due to (2.9). In view of Definition 2, this concludes the proof of (ii) for N larger than $c_N^* = 2[\frac{||\sigma||c_+}{\sigma_0 c_-}]^2$. This is because the quantity $||\phi||c_L$ approaches zero as $\delta \to 0$, so N can be improved arbitrarily close to c_N^*. When $L\phi \geq 0$ (sub-harmonic) then equality may be achieved (up to an integer value).

We verify (iii) from Lemma 1. We shall prove (iii) for boundary points ξ with $|I(\xi)| \leq N-1$, which includes the set $(\partial D \cap \partial D') \setminus A$. We note that, with probability one, only boundary points ξ with $I(\xi) = 1$ are visited. Abusing notation, we write $I(\xi) = i$ for the component located on the boundary ∂G. The process $\mathbf{y}(t)$ jumps if and only if a component reaches zero, which is equivalent to $\mathbf{x}(t)$ reaching ∂G^N at some point ξ (here we make use of the condition that $\phi(x) > 0$ except on A). To simplify notation, let $p_{Ij} = p_{ij}(\xi)$ denote the corresponding relocation probabilities.

Due to the condition in Definition 1 we have the non-random lower bound away from zero, uniformly in N:

$$(2.16) \quad \int_{G^N} \ln\Phi(\mathbf{x})\nu(\xi, d\mathbf{x}) - \ln\Phi(\xi) = \sum_{j\neq I} \frac{p_{Ij}}{2}\ln\Big(1 + \frac{\phi^2(x_j)}{\sum_{k\neq I}\phi^2(x_k)}\Big) \geq \frac{p_0}{2}\ln(\frac{N}{N-1}) > 0\,,$$

which shows (1.4) with $U = \frac{p_0}{2}\ln(\frac{N}{N-1})$. With the notation of Lemma 1, we have

$$(2.17) \qquad \forall \mathbf{x} \in D \qquad P_{\mathbf{x}}(l(D\setminus D') < \infty) = 1\,, \qquad P_{\mathbf{x}}(\alpha(D\setminus D') < \tau^*) = 1\,.$$

This concludes the proof of Step 1.

Step 2. For a $\delta > 0$ fixed, let F_k be the set of configurations with exactly $N-k$ particles in $\bar{G}_{2^k\delta}$ (or exactly k in the vicinity of the boundary $G \setminus \bar{G}_{2^k\delta}$). For a small $a > 0$,

$$F_k(a) = \{\mathbf{x} \in \bar{G}^N \mid \sum_{i=1}^{N} \mathbf{1}_{G\setminus\bar{G}_a}(x_i) = k\}\,, \qquad A_k(a) = \cup_{j=0}^{k} F_j(a)\,. \tag{2.18}$$

Let $F_k = F_k(2^k\delta)$ for $a = 2^k\delta$ and $A_k = \cup_{j=0}^{k} F_j$. We notice that $F_0 = (G_\delta)^N \subseteq \bar{D}_\delta$. Set $D' = F_N = (G \setminus \bar{G}_{2^N\delta})^N$, with $\alpha(D \setminus D')$ the first hitting time of $D \setminus D'$, as in Lemma 1. We have shown in *Step 1* that the lemma applies to the process $(\mathbf{x}(t))_{t\geq 0}$ and the open set D' and thus $P_{\mathbf{x}}(\alpha(D \setminus D') < \infty) = 1$ for all $\mathbf{x} \in D'$. In other words, if α_k is the first hitting time of A_k for all $k = 0, \ldots, N-1$, then $\alpha_{N-1} \leq \alpha(D \setminus D')$ is finite with probability one. To verify this inequality, we show that $\mathbf{x}(\alpha(D \setminus D')) \in A_{N-1}$. Since $\mathbf{x}(\alpha(D \setminus D')) \in F_N^c$ we only have to check that $F_N^c \subseteq A_{N-1}$.

$$F_N^c \subseteq A_{N-1}(2^N\delta) \subseteq A_{N-1}(2^{N-1}\delta) = \cup_{j=0}^{N-1} F_j(2^{N-1}\delta) \subseteq \cup_{j=0}^{N-1} A_j(2^j\delta) = A_{N-1}\,.$$

For all $k \geq 1$ and all $\mathbf{x} \in F_k$, $d(\mathbf{x}, F_0) \leq N2^N\delta$, $d(\mathbf{x}, \partial D) \leq 2^N\delta$, and thus $d(x, \partial(D \setminus F_0)) \leq N2^N\delta$, which implies that for any $\mathbf{x} \in F_k$, the time to reach either the interior set F_0 or the boundary ∂D is finite with probability one.

Let $\tau_0(D') = \alpha(D \setminus D')$ and $\tau_k(D')$, $k = 1, 2, \ldots, N-1$ be the first $N-1$ jump times coming right after $\alpha(D \setminus D')$. Starting with A_{N-1}, we want to reach A_{N-2}, ... A_0 with positive probability in each step. We proceed to show that for each $1 \leq k \leq N$ (in the proof k runs in decreasing order from $k = N$ to $k = 1$), the probability of the event $\mathcal{E} = \{\alpha_{k-1} \leq \tau_{N-k}(D')\}$ of reaching A_{k-1} at the time of the $(N-k)$-th jump or before has a lower bound away from zero, independent of the starting point in F_k. The fact that we reach the set at jump time is important, since we want to reach A_{k-1} at a time $\alpha_{k-1} < \tau^*$. Note first that $k = N$ is satisfied by *Step 1*. For other k, denote τ' the first time when one of the $N-k$ particles situated at time $t = 0$ in $G_{2^k\delta}$ reaches $G_{2^{k-1}\delta}$, $\mathcal{E}'$ the event that the first jump is onto one of these $N-k$ particles and τ'' the first time when one of the k particles in $G \setminus \bar{G}_{2^k\delta}$ at time $t = 0$ reaches ∂G. Then, for a fixed $T_0 > 0$,

$$\mathcal{E} \supseteqq \{\tau' > T_0,\, \tau'' \leq T_0\} \cap \mathcal{E}'\,. \tag{2.19}$$

Under the event from the right-hand side of (2.19) we have $\tau^D = \tau'' \leq T_0$, which implies that we may analyze all N particles independently up to τ^D-. At the same time, the jump is independent of past. The uniform lower bound for the probability of $\mathcal{E}$ is based on the bounds on the exit probability, respectively the redistribution probability ν_ξ when $k \leq N-1$

$$\inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}(\mathcal{E}) \geq \inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}\Big(\tau' > T_0\Big) \inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}\Big(\tau'' \leq T_0\Big) \inf_{\mathbf{x}\in \partial D\cap F_k} \nu_\xi(F_{k-1}) \tag{2.20}$$

$$\geq p_-(T_0, G_{2^k\delta}, G_{2^{k-1}\delta})^{N-k}\Big[1-(p_+(T_0,G,G))^k\Big]p_0 = p_{0,k} \tag{2.21}$$

where p_0 is the lower bound from Definition 1 and $p_\pm$ are defined in (2.5). Summarizing the information from (2.20)-(2.21), the probability to reach F_0 after the $N-1$ jumps following $\alpha(D\setminus D')$ when starting at an arbitrary $\mathbf{x}\in D\setminus D'$ has a positive lower bound $p = \Pi_{k=1}^{N-1} p_{0,N-k}$ independent of $\mathbf{x}$. With the notation $l(D_\delta)$ for the number of jumps until reaching the set $\overline{D_\delta}$, we have shown

$$\inf_{\mathbf{x}\in\overline{D'}} P_{\mathbf{x}}(l(D_\delta) \leq N-1) \geq p > 0\,. \tag{2.22}$$

Let $(X_n)_{n\geq 0}$ be the *interior chain* on D generated by $(\mathbf{x}(t))$ - see [11] for more details - displaying the consecutive positions of the process $(\mathbf{x}(t))$ at jumps times. In other words, $X_n = \mathbf{x}(\tau_n)$, $n\geq 0$. In discrete time $n = 0,1,\dots$ we denote $\alpha_X(B) = \inf\{n\geq 0 | X_n \in B\}$, B a Borel subset of D. We now apply Lemma 2 to $F = A_{N-1} \supseteq D\setminus F_N$, $\tau_X = \alpha_X(F_0)$, $m = N-1$ to show that $P_{\mathbf{x}}(\alpha_X(F_0) < \infty) = 1$ for all $\mathbf{x}\in D$. This shows that the number of jumps $l(\delta)$ until reaching $\bar{D}_\delta$ satisfies $P_{\mathbf{x}}(l(D_\delta) < \infty) = 1$, which implies that $P_{\mathbf{x}}(\alpha(D_\delta) < \tau^*) = 1$. Based on Lemma 3 we have that $\tau^* = \infty$ almost surely. □

Lemma 2. *Let $(X_n)_{n\geq 0}$ be a Markov chain on D, $F\subseteq D$ be a closed subset of D and τ_X a stopping time. If $P_{\mathbf{x}}(\alpha_X(F) < \infty) = 1$ for all $\mathbf{x}\in D$ and there exists an integer $m > 0$ and a number $p > 0$ independent of m such that $P_{\mathbf{x}}(\tau_X \leq m) \geq p$ uniformly in $\mathbf{x}\in F$, then $P_{\mathbf{x}}(\tau_X < \infty) = 1$ for all $\mathbf{x}\in D$.*

Proof. Let $\xi_0 = 0$, $\alpha_{X,1} = \inf\{n > \xi_0 \,|\, X_n \in F\}$, $\xi_1 = \alpha_{X,1} + m$ and inductively

$$\alpha_{X,l} = \inf\{n > \xi_{l-1} \,|\, X_n \in F\}\,, \quad \xi_l = \alpha_{X,l} + m\,, \qquad l \geq 2\,. \tag{2.23}$$

By construction, the stopping times ξ_l satisfy $P_{\mathbf{x}}(\xi_l < \infty)$ for all $\mathbf{x}\in D$ and $l = 1,2,\dots$, and $P_{\mathbf{x}}(\lim_{l\to\infty}\xi_l = \infty)$. Set k a positive integer. Successive applications of the strong Markov property on the intervals $[\xi_{l-1}, \alpha_l]$, $[\alpha_l, \xi_l]$, $l\geq 1$ give

$$P_{\mathbf{x}}(\tau_X > \xi_k) \leq E_{\mathbf{x}}[\Pi_{l=1}^k P_{X_{\alpha_{X,l}}}(\tau > m)] \leq (1-p)^k\,, \tag{2.24}$$

where the first inequality is obtained by neglecting the intervals $[\xi_{l-1}, \alpha_{X,l}]$. Since k is arbitrary, we proved that $P_{\mathbf{x}}(\tau_X < \infty) = 1$. $\square$

Lemma 3. *(Lemma 2 from [11]) Let $F \subseteq \bar{D}_\delta$ for some $\delta > 0$. If for any $\mathbf{x} \in D$ we have $P_{\mathbf{x}}(\alpha(F) < \tau^*) = 1$, then for any $\mathbf{x} \in D$ we have $P_{\mathbf{x}}(\tau^* = \infty) = 1$.*

3. Geometric ergodicity

In this section G is assumed bounded. We start by defining a special case of boundary regularity.

Definition 3. *If there exists an open set $G' \subseteq G$ and a function ϕ that, in addition to the properties from Definition 2, satisfy (i) $0 \leq \phi(x) \leq 1$ on $\bar{G}'$, (ii) $0 < \phi(x) < 1$ on G', $\phi(x) = 0$ on ∂G, (iii) $\phi(x) = 1$ on $\partial(G \setminus G')$, we say that G is ϕ - regular.*

Remark. We note that the solution to the boundary problem $L\phi = 0$ on G' with $\phi(x) = 0$ when $x \in \partial G$ and $\phi(x) = 1$ when $x \in \partial(G \setminus G')$ satisfies (i)-(iii) Definition 3 due to the maximum principle.

Proposition 1. *Suppose there exists ϕ as in Definition 3 with $G' \supseteq G \setminus \bar{G}_\delta$. Fix an index i, $1 \leq i \leq N$ and recall that $x_i(t)$ denotes the i - th component of $\mathbf{x}(t)$. If we denote by α_1 the first hitting time of the set $\bar{G}_\delta$ by the process $(x_i(t))$, then there exist $\theta > 0$, $C_0 > 0$ independent of $\mathbf{x} \in G'$ such that $E_{\mathbf{x}}[\exp(\theta \alpha_1)] \leq C_0$.*

Proof. Denote $y_i = \phi(x_i)$, $1 \leq i \leq N$ and the process $(\mathbf{y}(t))$ with components $y_i(t) = \phi(x_i(t))$, $t \geq 0$. In the following the particle index i is not important and we denote y_i simply by y and similarly x_i by x. Denote by β_1 the first hitting time of the point $y = 1$ by the process $(y(t))$. We have the almost sure inequality $\alpha_1 \leq \beta_1$.

The process $(y(t))$ evolves in $[0, 1]$ undergoing jumps at a subset of the jump times (τ_l) for the process $(\mathbf{x}(t))$. To simplify notation, we shall still denote these jumps by τ_l, $l \geq 1$, $\tau_0 = 0$. Due to the properties of ϕ, with probability one, at each time τ_l, the jump pushes the one-dimensional process $y(t)$ to the right, from $y(\tau_l-) = 0$ to $y(\tau_l) > 0$. We shall construct by coupling a new process $z(t)$ evolving on $(-\infty, 1]$ with a monotonicity property. At start, the processes $z(t)$ and $y(t)$ coincide - until τ_1. At τ_1, $z(t)$ suppresses the jump, but continues to diffuse being driven by the same stochastic differential equation as $y(t)$. Based on (2.4), we construct inductively for $l \geq 0$ a sequence $z_{0,l}$, by setting $z_{0,0} = y_0 = \phi(x_0)$, and a process

$$dz(t) = \tilde{b}_i(t)dt + \tilde{\sigma}_i(t)dW(t)\,, \qquad \tau_l \le t < \tau_{l+1}\,, \quad z(\tau_l) = z_{0,l}\,, \tag{3.1}$$

where the coefficients are defined in (2.8). At each step, we update $z_{0,l+1} := z(\tau_{l+1}-)$. Due to the pathwise coupling (3.1), $z(t) \le y(t)$ almost surely when $z(\tau_l) \le y(\tau_l)$, which is true by construction. Denoting with γ_1 the first hitting time of the point one by $(z(t))$, we see that $\beta_1 \le \gamma_1$ with probability one. For $z \in [0,1]$ the starting point $z = \phi(x)$ and for $\theta > 0$, we have,

$$E_{\mathbf{x}}[\exp(\theta\alpha_1)\,|\,(\mathbf{x}(0))_i = z] \le E_{\mathbf{x}}[\exp(\theta\gamma_1)\,|\,(\mathbf{x}(0))_i = z]\,. \tag{3.2}$$

Since the drift $\tilde{b}_i(t) = L\phi(x_i(t))$ is uniformly bounded, the Cameron-Girsanov's formula reduces the question of the upper bound of the left hand side of (3.2) to the case of a continuous martingale with uniformly bounded quadratic variation from the right hand side. If $||\nabla\phi(z)||$ is bounded away from zero, a time change shows that the right hand side of (3.2) is bounded above as soon as it is finite for a Brownian motion with finite negative drift, which is immediate. □

Theorem 2. *Assume G is bounded and there exists a function satisfying the conditions of Definition 3. Then, provided N is sufficiently large such that the process be non-explosive, then $(\mathbf{x}(t))$ is geometrically ergodic. The invariant probability measure has a density with respect to the Lebesgue measure equal, modulo a normalizing constant, to the integral of the Green function of L with Dirichlet boundary conditions on G with respect to the invariant probability measure of the interior chain (X_n).*

Remark. We refer the reader to Theorem 3 in [11] for more details on the invariant measure. In the context of the Fleming-Viot particle process, obtaining (3.4) needs the intermediate step from Proposition 1.

Proof. The set $\overline{D_\delta}$ is a *small* set for the process due to the fact that $(\mathbf{x}(t))$ has a density bounded below by the density function of the process killed at the boundary; in its turn, this density function has a uniform lower bound on $\overline{D_\delta}$ for any $t > 0$. Exponential ergodicity is guaranteed [4] by the sufficient condition (3.4) that there exists an exponential moment of the time to reach $\overline{D_\delta}$, uniformly over all $\mathbf{x} \in D = G^N$.

Most of the proof is contained in Theorem 3 in [11]. We prove the part that is new to the context of the Fleming-Viot redistribution function. Recall that $D = G^N$, $D' = (G\backslash\overline{G_{2^N\delta}})^N$

and $\alpha(D \setminus D')$ is the first exit time from D', i.e. the hitting time of the set of configurations with at least one particle at distance larger than $2^N\delta$ from the boundary. Proposition 1 shows that there exists $\theta > 0$ such that

$$\sup_{\mathbf{x} \in D} E_{\mathbf{x}}[e^{\theta\alpha(D \setminus D')}] < \infty \tag{3.3}$$

due to the uniform bound and a Markov property inductive argument similar to the one in Lemma 2. We want a similar uniform bound on $\alpha(D_\delta)$. This is guaranteed by the Step 2 of the proof of Theorem 1, where it is shown that once in $D \setminus D'$, the probability to reach $\overline{D_\delta}$ in $N-1$ consecutive jumps in time at most T (for a fixed but arbitrary T) is bounded away from zero uniformly on the configuration in $D \setminus D'$. Another iteration of the argument from Lemma 2 in continuous time setting (there is virtually no modification needed) gives

$$\sup_{\mathbf{x} \in D} E_{\mathbf{x}}[e^{\theta\alpha(D_\delta)}] < \infty \tag{3.4}$$

concluding the proof. □

4. **Examples of sets satisfying the regularity conditions**

The following examples assume G is a bounded, regular domain, G' be a vicinity of the boundary ∂G in the sense that there exists $\delta > 0$ such that $G \setminus \bar{G}' \subseteq G_\delta$ and $x' \in G \setminus \bar{G}'$.

Proposition 2. *Suppose G is bounded with the interior sphere condition. If*

(i) the Green function $K(\cdot, x') \in C^1(\bar{G}')$, set $\phi(x) = K(x, x')$ on G' and extend it continuously over G; or, alternatively

(ii) the first eigenfunction $\phi_0(x)$ of the operator L on G with Dirichlet boundary conditions is continuous and has continuous derivatives up to the boundary, set $\phi(x) = \phi_0(x)$,

then Definition 2 holds for $\phi(x)$ in each case. Moreover, a sufficient conditions for both (i) and (ii) is $\partial G \in C^2$.

Proof. The Green function satisfies $LK(x, x') = 0$ in G', is positive in G', vanishes on ∂G. Due to the smoothness of the boundary $\partial G \in C^2$ or directly from assumptions (i) and (ii), $\phi \in C^1(\bar{G})$. The Hopf maximum principle [8] shows that $\langle \nabla\phi(x), n \rangle < 0$ on ∂G, where n is the outward normal to ∂G. From the boundedness of the domain, G and ∂G are compact, and from the continuity up to the boundary we have that $||\nabla\phi(x)||$ is bounded away from zero in a neighborhood of the boundary (otherwise it would reach zero on ∂G). For sufficiently small δ we obtain all conditions required. □

Proposition 3. *Let G be a regular domain, G' be a vicinity of the boundary ∂G satisfying the interior sphere condition, and $\phi(x)$ be the solution of the Dirichlet boundary problem on G' for L with $\phi(x) = 0$ on ∂G and $\phi(x) = 1$ on $\partial G' \setminus \partial G$. If ϕ has continuous derivatives up to ∂G, then Definition 3 is satisfied. A sufficient conditions is $\partial G \in C^2$.*

Proof. Using the strong maximum principle we obtain that $0 < \phi(x) < 1$ on G'. □

5. **The immortal particle**

This section investigates the particle ancestry. The realization of the process is a tree with continuous branches, representing diffusive episodes performed by the particles. Reaching the boundary ends a certain branch, that will never be revived. Branching at a given location allows the continuation of the tree, provided non-extinction (Theorem 3), ad infinitum. The goal is to prove that, almost surely, there exists a unique infinite continuous path on the tree, in the sense of Theorem 3 (iv). This is, informally, the immortal particle. It is not a proper tagged particle because it changes its label infinitely many times.

The reader is reminded that $x_i(t)$ represents the particle of *index* $i \in \{1, \ldots, N\}$ and that the indices are fixed forever; also, $(\tau_l)_{l\geq 0}$, $\tau_0 = 0$ denote the increasing sequence of times when particles hit the boundary. At time $t = 0$, each particle is given a label (or color). The label is preserved as long as the particle is alive; when it is killed, the particle that replaces it will acquire the label of the particle it jumps to. Or, in a different but equivalent interpretation, the particle is killed and the newly born particle will have the same label as its parent. We want to show that, with probability one, exactly one label survives. Ultimately, all particles at time t can be traced to only one original ancestor, all other lineages (to be defined precisely) dying in finite time.

5.1. **The multi-color process.**

Formally we shall consider a Markov process with state space $(G \times \mathcal{C})^N$, where $\mathcal{C}$ is a finite set of labels (colors). One example is $\mathcal{C} = \{1, \ldots, N\}$ and another important one is when $\mathcal{C} = \{0, 1\}$. It will be shown that the two-color model is sufficient to trace ancestry. An element in the state space is a vector with N components $(x_i, \mathcal{C}(x_i))$, $1 \leq i \leq N$ designating the position x_i of particle i and its color. We used $\mathcal{C}(x_i) \in \mathcal{C}$ for the color of particle to avoid more complicated notation.

The particles $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t)) \in G^N$ follow exactly the branching mechanism from Section 2 with redistribution measure (2.1). At the same time, the labels follow the rule that they remain constant until the particle hits the boundary, at which time it instantaneously and always adopts the label of the particle it jumped to; equivalently, the particle reaching the boundary is killed and a new particle is born from a surviving one, with the same label as the parent. Naturally the latest interpretation is more relevant to our investigation. It is easy to see that the joint process (particle-label) is Markovian.

Proposition 4. *Assuming the unlabeled process is non-explosive, with probability one, all but one label have finite lifetime.*

Remark. 1) Once only one color has been achieved, it is evident that the process follows the unlabeled branching mechanism and continues its evolution forever (as long as the process is not explosive).

2) Considering a discrete space and time version of the process, the reader may see why the proposition is true, since all multi-colored states are transient. It is sufficient to observe that one color can be *forced* to hit the boundary while all other colors are not reaching the boundary and upon killing only the other colors are allowed to branch (a small but positive probability event).

Proof. The proof follows a different idea than described in Remark 2), better suited to the context of diffusions. First, we notice that it is enough to prove the proposition for two colors (zero and one) in the sense that the time for one color to disappear will be shown to be finite almost surely. At time zero we re-label particles of a type with one and all the others with zero. Inductively, it will follow that the number of colors is reduced to exactly one in finite time. Denote τ_L the first time when the number of labels has been reduced to one, with the usual convention that $\tau_L = \infty$ if the event does not happen in finite time.

Let $\delta > 0$ be such that $\bar{G}_{2\delta} \subset G$ (the reason why we use 2δ becomes apparent immediately). On the one hand, we know that from any initial position $\mathbf{x}$, the particle system will reach the complement $F_{2\delta}$ of $(G \setminus G_{2\delta})^N$ a.s., that is, at least one particle will be within $\bar{G}_{2\delta}$. On the other hand, for $T > 0$ fixed and $\mathbf{x} \in F_{2\delta}$, we shall obtain a lower bound $p_0 > 0$ of $P_{\mathbf{x}}(\tau_L \leq T)$, uniformly over $\mathbf{x} \in F_{2\delta}$. Starting with an arbitrary $\mathbf{x}$, the system will have an infinite number of attempts to reach a one-label configuration. Since the failure probability is $1 - p_0 < 1$ in each episode, it follows that $\tau_L < \infty$ with probability one.

Part 1. Let $\mathbf{x} \in F_{2\delta}$. Without loss of generality we assume that $x_1 \in \bar{G}_{2\delta}$. Let $K = \{\tau^{\bar{G}_\delta,1} > T\}$, where $\tau^{\bar{G}_\delta,1}$ is the first time when the particle #1 hits $G \setminus G_\delta$, $\tau_1^{G,j}$, $\tau_2^{G,j}$ the first, respectively second boundary hit of particle $\#j$, $1 \leq j \leq N$. Denote A_j, B_j, C_j the events pertaining to particles $\#j$, $2 \leq j \leq N$

$$A_j = \{\tau_1^{G,j} \leq T\}\,, \quad B_j = \{x_j(\tau_1^{G,j}) = x_1(\tau_1^{G,j})\}\,, \quad C_j = \{\tau_2^{G,j} > T\} \tag{5.1}$$

with $A = \cap_{j=2}^N A_j$, $B = \cap_{j=2}^N B_j$ and $C = \cap_{j=2}^N C_j$. In other words, K means that x_1 will not exit G_δ before time T; A_j that x_j hits the boundary in $[0,T]$; B_j that x_j jumps to the location of x_1 at its first boundary hit, and C_j that x_j will not jump again before time T. With the observation that $\{\tau_L \leq T\} \supseteq A \cap B \cap C \cap K$, it is sufficient to prove $P_{\mathbf{x}}(A \cap B \cap C \cap K) \geq p_0 > 0$ with p_0 independent of $\mathbf{x} \in F_{2\delta}$. Two particles are independent until they meet, i.e. there is a jump/birth involving the two. Consequently, conditional on K, the events $(A_j \cap B_j \cap C_j)_{2 \leq j \leq N}$ are mutually independent with

(5.2)

$$P_{\mathbf{x}}(A \cap B \cap C \cap K) = P_{\mathbf{x}}(A \cap B \cap C \,|\, K) P_{\mathbf{x}}(K) = \Pi_{j=2}^N P_{\mathbf{x}}(A_j \cap B_j \cap C_j \,|\, K) P_{x_1}(\tau^{\bar{G}_\delta,1} > T)$$

$$\geq \Pi_{j=2}^N P_{\mathbf{x}}(A_j \cap B_j \cap C_j \,|\, K) p_-(T, G_{2\delta}, G_\delta)\,, \tag{5.3}$$

where $p_\pm$ are defined in (2.5). We write

$$P_{\mathbf{x}}(A_j \cap B_j \cap C_j \,|\, K) = P_{\mathbf{x}}(C_j \,|\, A_j \cap B_j \cap K) P_{\mathbf{x}}(A_j \cap B_j \,|\, K) \tag{5.4}$$

and see that the first factor is bounded below (by introducing $\tau_2^{G,j} > T + \tau_1^{G,j}$ instead of $\tau_2^{G,j} > T$) by

$$P_{\mathbf{x}}(C_j \,|\, A_j \cap B_j \cap K) \geq \int_G P_x(\tau^G > T) P_{\mathbf{x}}(x_j(\tau_1^{G,j}) \in dx \,|\, A_j \cap B_j \cap K) \geq p_-(T, G_\delta, G) \tag{5.5}$$

(note that the position of the jump is on the trajectory of x_1 that stays in G_δ). At the same time A_j, B_j and K are independent with $P_{\mathbf{x}}(A_j|K) = P_{\mathbf{x}}(A_j) \geq 1 - p_+(T, G_{2\delta}, G)$ and $P_{\mathbf{x}}(B_j|K) = (N-1)^{-1}$. Putting all together, the probability from (5.2) is bounded below by

$$p_0 = \Big[p_-(T, G_\delta, G)(1 - p_+(T, G_{2\delta}, G))(N-1)^{-1}\Big]^{N-1} p_-(T, G_{2\delta}, G_\delta) > 0\,. \tag{5.6}$$

Part 2. We shall apply Lemma 2 with $F = F_{2\delta}$, $\tau = \tau_L$ to obtain the conclusion of the theorem. $\square$

Let $l : [0, \infty) \to \{1, 2, \ldots, N\}$ and $\eta : [0, \infty) \to \bar{G}$ be random processes adapted to $(\mathcal{F}_t)_{t \geq 0}$ such that (i) $l(t)$ is piecewise constant and $\eta(t) = x_{l(t)}(t)$ on intervals $[\tau_{k-1}, \tau_k)$, $k \geq 1$ and (ii) η continuous with $\eta(t) \equiv \eta(\tau_k -)$ for all $t \geq \tau_k$ if $\eta(\tau_k -) \in \partial G$. A pair $(l(\cdot), \eta(\cdot))$ is said a *lineage*. The stopping time τ_k when (ii) happens is said the *lifetime* of the lineage and is denoted by $\tau(\eta)$.

For $t_1 < t_2$, i_1, i_2 two of the N labels, we say that $x_{i_1}(t_1)$ is an *ancestor* of $x_{i_2}(t_2)$ (or there exists a lineage from $x_{i_1}(t_1)$ to $x_{i_2}(t_2)$) and we write $(t_1, i_1) \preceq (t_2, i_2)$ if there exists a lineage $(l(\cdot), \eta(\cdot))$ with $\tau(\eta) \geq t_2$ such that $l(t_1) = i_1$, $\eta(t_1) = x_{i_1}(t_1)$ and $l(t_2) = i_2$, $\eta(t_2) = x_{i_2}(t_2)$. On the set of pairs (t, i), the lineage introduces a relation of partial order.

Theorem 3. *Assume G is a regular bounded domain and the process is non-explosive. Let $t_1 < t_2$ and i_1, i_2 two of the N labels. If $(t_1, i_1) \preceq (t_2, i_2)$, then*

(i) the lineage they belong to is unique up to time $t = t_2$;

(ii) the labels/colors are identical at both endpoints, $\mathcal{C}(x_{i_1}(t_1)) = \mathcal{C}(x_{i_2}(t_2))$ and as a consequence, a lineage will never change label;

(iii) For any $t \geq 0$ and any index i, there exists an index i_0 such that $(0, i_0) \preceq (t, i)$;

(iv) There exists a unique lineage with infinite lifetime.

Proof. (i) Assume $(l'(\cdot), \eta'(\cdot))$, $(l''(\cdot), \eta''(\cdot))$ are two lineages going from (t_1, i_1) to (t_2, i_2). Lineages may intersect in two ways: either on open intervals (τ_{k-1}, τ_k) as diffusion paths (with zero probability except in dimension one), or at branching times τ_k. Only intersections of the second type are *proper* because the particles do not interact during the diffusive episodes. Two lineages will properly intersect at time t only if they coincide on $[0, t]$; otherwise, they will have to intersect in the open set G, which is impossible by construction. Evidently, lineages may diverge after t.

(ii) The colors may change only at times τ_k. At jump time, the particle performing the jump from the boundary adopts the label of the one in G, whose label coincides with the label of the lineage. Again by construction, at a branching point the label is preserved for all offspring, so the lineage does not change label, having $\mathcal{C}(x_{l(\tau_k -)}) = \mathcal{C}(x_{l(\tau_k)})$.

(iii) Theorem 1 shows that $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ and $\lim_{k \to \infty} \tau_k = +\infty$ a.s. Let $k(t)$ be the integer $k \geq 1$ such that $\tau_{k-1} \leq t < \tau_k$; then one can verify (iii) by induction over k.

(iv) At time $t = 0$ we label $\mathcal{C}(x_i(0)) = i$ for all indexes i. We know from Proposition 4 that $\tau_L < \infty$ a.s., which implies due to (ii) that at time $t = \tau_L$ only one lineage, starting at $(0, i_0)$ is still alive (did not reach the boundary). Due to (iii), we deduce that at time $t \geq \tau_L$, all particles have lineages all the way to $(0, i_0)$. Let τ_L^k, $k \geq 1$ be defined inductively by setting $\tau_L = \tau_L^1$ and re-labeling the particles at time τ_L by $\mathcal{C}(x_i(\tau_L)) = i$ with $\tau_L^2 > \tau_L^1$ being exactly the time after τ_L^1 when all labels become identical once again. Due to the strong Markov property and again Proposition 4, $\tau_L^2 < \infty$ a.s. and we re-apply (ii)-(iii) to see that only one index i_1 survives, making (τ_L^1, i_1) the only ancestor of all (τ_L^2, i), $1 \leq i \leq N$. Since $\tau_L \geq \tau_1$ we immediately have τ_L^k bounded below by a subsequence of $(\tau_{j_k})_{k\geq 1}$ of the boundary hits. Then $\lim_{k\to\infty} \tau_L^k = +\infty$ with probability one, implying that the construction can be done for any $t > 0$. The uniqueness is a consequence of (i). $\square$

6. **The two particle case**

As seen in Section 2, the many particle case is always non-explosive, which points to $N = 2$ as a benchmark of critical behavior. Here we can derive the transition function of the surviving particle. Denote X the position of the surviving particle at the time of the first boundary visit. If the particles start at x_1 and x_2 respectively, then

$$P_{(x_1,x_2)}(X \in dy) = P_{x_1}(x_1(\tau_2) \in dy, \tau_1 > \tau_2) + P_{x_2}(x_2(\tau_1) \in dy, \tau_2 > \tau_1) \tag{6.1}$$

$$= \int_0^\infty P_{x_1}(x_1(t) \in dy, \tau_1 > t) P_{x_2}(\tau_2 \in dt) + \int_0^\infty P_{x_2}(x_2(t) \in dy, \tau_2 > t) P_{x_1}(\tau_1 \in dt)\,. \tag{6.2}$$

When $x_1 = x_2 = x$ we obtain the transition probability $S(x, dy)$ of the interior Markov chain tracing the locations $X_k = x_1(\tau_k)$, $k \geq 1$ *right after a jump*. It is

$$S(x, dy) = P(X_1 \in dy \,|\, X_0 = x) = P_x(X \in dy) = 2\int_0^\infty P^G(t, x, dy) P_x(\tau^G \in dt)\,, \tag{6.3}$$

where

$$P_x(\tau^G > t) = \int_G p^G(t, x, y) dy\,. \tag{6.4}$$

Combining (6.3) and (6.4) and integrating by parts we can write the alternative formula (not used in this paper)

$$P_x(X \in dy) = 2\delta_x(dy) + 2\int_0^\infty P^G(\tau^G > t) \partial_t p^G(t, x, dy) dt\,. \tag{6.5}$$

Due to independence,

$$P_x(\tau_1 \wedge \tau_2 > t) = (P_x(\tau^G > t))^2 \,, \qquad E_x[\tau_1 \wedge \tau_2] = \int_0^\infty (P_x(\tau^G > t))^2 dt \,. \tag{6.6}$$

6.1. **Two particles on the half-line.** Assume $D = (0, \infty)$, $N = 2$ and each particle follows $x_i(t) = x_i - \mu t + w_i(t)$, $i = 1, 2$, where $w_i(t)$ are independent Brownian motions. The density function of the Brownian motion on the positive half-line with drift $-\mu$ killed at the origin is

$$p^G(t, x, y) = \frac{1}{\sqrt{2\pi t}}\Big(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}\Big)e^{-\mu(y-x)-\frac{1}{2}\mu^2} \,, \tag{6.7}$$

as can be seen by applying Girsanov's formula or directly by verification of the Kolmogorov equations. Starting with (6.4) and noticing that the adjoint of L is $L_y^* = \frac{1}{2}\frac{d^2}{dy^2} + \mu\frac{d}{dy}$ with Dirichlet b.c. at zero, the density of τ^G, in this case, is

$$\frac{d}{dt}P_x(\tau^G \in dt) = -\int_G \frac{d}{dt}p^G(t, x, y)dy = -\int_G L_y^* p^G(t, x, y)dy \tag{6.8}$$

$$= \frac{1}{2}\partial_y p^G(t, x, 0) \,. \tag{6.9}$$

The transition probability (6.3) reads

$$P_x(X \in dy) = \int_0^\infty P^G(t, x, dy)\partial_y p^G(t, x, 0)dt \,. \tag{6.10}$$

Proposition 5. *The following estimates are satisfied*

$$2E_x[\tau_1 \wedge \tau_2] = E_x[X^2] \sim o(x) \,, \qquad \lim_{x\to 0} \frac{E_x[X]}{x} = 2 \,. \tag{6.11}$$

Proof. Observing that $-\mu < 0$, then $\tau^G < \infty$ and even more so $\tau_1 \wedge \tau_2 \le \tau^G < \infty$ with probability one, the optional stopping theorem (at $t = \tau_1 \wedge \tau_2$) applied to the martingales $M_1(t) = x_1(t) + x_2(t) + 2\mu t$ and $M_2(t) = x_1^2(t) + x_2^2(t) - 2x_1(t)x_2(t) - 2t$ shows that

$$E_x[X] + 2\mu E_x[\tau_1 \wedge \tau_2] = 2x \,, \qquad E_x[X^2] - 2E_x[\tau_1 \wedge \tau_2] = 0 \,. \tag{6.12}$$

We want to prove the two limits (the second is a consequence of the first)

$$\lim_{x\to 0} \frac{2E_x[\tau_1 \wedge \tau_2]}{x} = \lim_{x\to 0} \frac{E_x[X^2]}{x} = 0 \,, \qquad \lim_{x\to 0} \frac{E_x[X]}{x} = 2 \,. \tag{6.13}$$

Since we calculate the limit as $x \to 0$, we may assume $0 < x \le 1$. Using (6.6), we shall prove directly the first limit in (6.13)

$$\lim_{x\to 0} \frac{\int_0^\infty (P_x(\tau^G > t))^2 dt}{x} = \lim_{x\to 0} \Big(2\int_0^\infty P_x(\tau^G > t)\frac{d}{dx}P_x(\tau^G > t)dt\Big) = 0 \,. \tag{6.14}$$

To have (6.14), we use L'Hospital's rule; it is necessary to justify the differentiation under the integral and the limits as $x \to 0$.

From (6.7) we derive

$$P_x(\tau^G > t) = \Phi(\frac{x-\mu t}{\sqrt{t}}) - e^{2\mu x}(1 - \Phi(\frac{x+\mu t}{\sqrt{t}}))\,, \tag{6.15}$$

where $\Phi'(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$. This is evidently in the interval $[0,1]$ and thus bounded and has limit zero at $x = 0$. It remains to show that the absolute value of the derivative has an upper bound, uniformly in $x \in [0,1]$ that is integrable in $t \in (0,\infty)$. The derivative is

$$\frac{d}{dx}P_x(\tau^G > t) = \frac{1}{\sqrt{t}}\Big(\Phi'(\frac{x-\mu t}{\sqrt{t}}) + e^{2\mu x}\Phi'(\frac{x+\mu t}{\sqrt{t}})\Big) - 2\mu e^{2\mu x}\Big(1 - \Phi(\frac{x+\mu t}{\sqrt{t}})\Big)\,. \tag{6.16}$$

We break down (6.16) in the term containing $\frac{1}{\sqrt{t}}\Phi'(\frac{x-\mu t}{\sqrt{t}})$; the term containing $\frac{e^{2\mu x}}{\sqrt{t}}\Phi'(\frac{x+\mu t}{\sqrt{t}})$, both bounded above by $\frac{e^{\mu}}{\sqrt{t}}\Phi'(\mu\sqrt{t})$, which is integrable in t on $(0,\infty)$; and the third part, with absolute value bounded above by $2\mu e^{2\mu}(1-\Phi(\mu\sqrt{t}))$, which is also integrable

$$\int_0^\infty 1 - \Phi(\mu\sqrt{t})dt \le \Big(1 + \sqrt{\frac{2}{\pi}}\Big)\frac{1}{\mu^2} < \infty\,.$$

The last inequality comes from the estimate on the error function

$$1 - \Phi(\mu\sqrt{t}) = \int_{\mu\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \le \int_{\mu\sqrt{t}}^\infty z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2 t}{2}}$$

when $\mu\sqrt{t} \ge 1$. □

6.2. **Brownian motion without drift.**

Proposition 6. *When $\mu = 0$, the distribution of $V = X/x$ is independent of the starting point x having density*

$$f_V(v) = \frac{8v}{\pi[(v-1)^2+1][(v+1)^2+1]}\,. \tag{6.17}$$

Since $f_V(v) \sim O(v)$ at $v = 0$ and $f_V(v) \sim O(v^{-3})$ at $v = +\infty$, the random variable V has moments $E[V^a]$ up to $a < 2$, with $\mu_V = 2$, $\sigma_V^2 = \infty$ and $E[\ln V] > 0$.

Proof. The cumulative distribution function of the hitting time τ^G, based on (6.4) applied to (6.7) is $2(1 - \Phi(\frac{x}{\sqrt{t}}))$ and the density is

$$-\frac{d}{dt}P_x(\tau^G > t) = \frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}} \tag{6.18}$$

so (6.3) reads

$$\frac{P_x(X \in dy)}{dy} = \int_0^\infty \frac{x}{\pi t^2}\Big(e^{-\frac{(y-x)^2+x^2}{2t}} - e^{-\frac{(y+x)^2+x^2}{2t}}\Big)dt \tag{6.19}$$

$$= \frac{x}{\pi}\Big(\frac{2}{(y-x)^2+x^2} - \frac{2}{(y+x)^2+x^2}\Big) = \frac{1}{x} f_V(\frac{y}{x})\,. \tag{6.20}$$

In the last equality we identified the alternative formula

$$f_V(v) = \frac{2}{\pi}\Big(\frac{1}{(v-1)^2+1} - \frac{1}{(v+1)^2+1}\Big) \tag{6.21}$$

with

$$F_V(v) = P(V \le v) = 1 - \frac{2}{\pi}\Big(\arctan(v+1) - \arctan(v-1)\Big)\,. \tag{6.22}$$

One can calculate explicitly

$$E[V] = \Big[\frac{1}{\pi}\ln(\frac{1+(v-1)^2}{1+(v+1)^2}) + \frac{2}{\pi}(\arctan(v-1) + \arctan(v+1))\Big]\Big|_0^\infty = 2\,. \tag{6.23}$$

The logarithm $\ln V$ is integrable and we can determine numerically that $E[\ln V] \approx 0.34$.

$\square$

The interior chain (X_n) satisfies $\ln X_n = \ln x_0 + \sum_{k=1}^n \ln V_k$ where V_k are i.i.d. with distribution (6.17). By the law of large numbers, we have $\frac{\ln X_n}{n} \to E[\ln V] > 0$ as $n \to \infty$ with probability one so $P_{x_0}(\lim_{n\to\infty} X_n = \infty) = 1$.

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