Cyclic Sieving of Multisets with Bounded Multiplicity and the Frobenius Coin Problem

Séminaire Lotharingien de Combinatoire 93, Pocinho

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1. Generalized Binomial Coefficients

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Two interpretations of binomial coefficients:

$$E(n;t) = \prod_{i=1}^{n}(1+t) = \sum_{k\geq 0}\binom{n}{k}t^k,$$

$$H(n;t) = \prod_{i=1}^{n}(1+t+t^2+\cdots) = \sum_{k\geq 0}\binom{n+k-1}{k}t^k.$$

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We consider the following interpolation:

$$H^{(b)}(n;t) = \prod_{i=1}^{n}(1+t+\cdots+t^{b-1}) = \sum_{k\geq 0}\binom{n}{k}^{(b)}t^k.$$

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Note that $\binom{n}{k}^{(2)} = \binom{n}{k}$ and $\binom{n}{k}^{(b)} = \binom{n+k-1}{k}$ when $b > k$. We will write

$$\binom{n}{k}^{(\infty)} = \binom{n+k-1}{k}.$$

Example: $n = 3$ and $b = 4$. The generating function is

$$\begin{aligned} H^{(4)}(3;t) &= (1+t+t^2+t^3)^3 \\ &= 1+3\,t+6\,t^2+10\,t^3+12\,t^4+12\,t^5+10\,t^6+6\,t^7+3\,t^8+t^9. \end{aligned}$$

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The coefficients are

k	0	1	2	3	4	5	6	7	8	9
$\binom{3}{k}^{(4)}$	1	3	6	10	12	12	10	6	3	1

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k	0	1	2	3	4	5	6	7	8	9
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In general we have $\binom{n}{k}^{(b)} = 0$ for $(b-1)n > k$ and

$$\binom{n}{0}^{(b)} + \binom{n}{1}^{(b)} + \cdots + \binom{n}{(b-1)n}^{(b)} = b^n.$$

Remark: $\binom{n}{k}^{(b)} / b^n$ is the probability of getting a sum of k in n rolls of a fair b-sided die with sides labeled $\{0, 1, \ldots, b-1\}$.

1. Generalized Binomial Coefficients

Remarks:

- The numbers $\binom{n}{k}^{(b)}$ occur often but they don't have a standard name.
- We roughly follow Euler's (1778) notation: $\left(\frac{n}{k}\right)^{b}$.
- Belbachir and Igueroufa (2020) compiled a historical bibliography.

2. Symmetric Functions and q-Analogues

Recall the generating functions for *elementary* and *complete* symmetric polynomials:

$$E(z_1, \ldots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t) = \sum_{k\geq 0} e_k(z_1, \ldots, z_n)t^k,$$

$$H(z_1, \ldots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t + (z_i t)^2 + \cdots) = \sum_{k\geq 0} h_k(z_1, \ldots, z_n)t^k.$$

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$$H^{(b)}(z_1, \dots, z_n; t) = \prod_{i=1}^{n}(1 + z_i t + \cdots (z_i t)^{b-1}) = \sum_{k \geq 0} h_k^{(b)}(z_1, \dots, z_n) t^k.$$

Note that $h_k^{(2)} = e_k$ and $h_k^{(b)} = h_k$ when $b > k$. We will write $h_k^{(\infty)} = h_k$.

We can view $h_k^{(b)}(z_1, \ldots, z_n)$ as a generating function for lattice points in a diagonal slice of the integer box $\{0, 1, \ldots, b-1\}^n$:

$$X := \{(x_1, \ldots, x_n) \in \{0, 1 \ldots, b-1\}^n : x_1 + x_2 + \cdots + x_n = k\}.$$

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Then we have

$$h_k^{(b)}(z_1, \dots, z_n) = \sum_{\mathbf{x} \in X} \mathbf{z}^{\mathbf{x}} = \sum_{\mathbf{x} \in X} z_1^{x_1} z_2^{x_2} \cdots z_n^{x_n}.$$

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We can also view these lattice points as k-multisubsets of $\{1, 2, \ldots, n\}$ with multiplicities bounded above by b:

$$(x_1, x_2, \ldots, x_n) \quad \longleftrightarrow \quad \{\underbrace{1, \ldots, 1}_{x_1 \text{ times}}, \underbrace{2, \ldots, 2}_{x_2 \text{ times}} \ldots, \underbrace{n, \ldots, n}_{x_n \text{ times}}\}.$$

$$b = 2: \quad k\text{-subsets of } \{1, \ldots, n\},$$

$$b = \infty: \quad k\text{-multisubsets of } \{1, \ldots, n\}.$$

2. Symmetric Functions and q-Analogues

Example: $n = 3$ and $k = 3$ for various values of b:

			030			
		120		021		
	210		111		012	
300		201		102		003

$$h_3^{(2)}(z_1, z_2, z_3) = z_1 z_2 z_3,$$
$$h_3^{(3)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2,$$
$$h_3^{(4)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_2^3,$$
$$h_3^{(5)}(z_1, z_2, z_3) = z_1 z_2 z_3 + z_1^2 z_2 + \cdots + z_2 z_3^2 + z_1^3 + z_2^3 + z_2^3,$$
$$\vdots$$

A natural q-analogue of $\binom{n}{k}^{(b)}$ is given by the principal specialization of $h_k^{(b)}$:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} := h_k^{(b)}(1, q, \ldots, q^{n-1}) = \sum_{\mathbf{x} \in X} q^{0x_1 + 1x_2 + 3x_2 + \cdots + (n-1)x_n}.$$

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This generalizes the standard q-binomial coefficients in the following sense:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(2)} = q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(\infty)} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q.$$

Opinion: This is the reason why sometimes we multiply $\begin{bmatrix} n \\ k \end{bmatrix}_q$ by $q^{k(k-1)/2}$ and sometimes we don't.

Example: $n = 3$ and $k = 3$ for various values of b:

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			030			
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$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(2)} = q^3,$$

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(3)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5,$$

$$\begin{bmatrix}3\\3\end{bmatrix}_q^{(\infty)} = q^3 + q^1 + 2q^2 + 2q^4 + q^5 + 1 + q^3 + q^6.$$

Remarks:

- Like the numbers $\binom{n}{k}^{(b)}$, the polynomials $h_k^{(b)}(z_1, \ldots, z_n)$ don't have a standard name or notation.

- Doty and Walker (1992) used $h'_k(n)$ and called them *modular complete symmetric polynomials*.

- Fu and Mei (2020) used $h_k^{[b-1]}$ and called them *truncated complete*.

- Grinberg (2022) used $G(b, k)$ and called them *Petrie symmetric functions*. He now regrets this name (personal communication).

- Since the definition is simple I believe that the name should be simple. In the paper I called them b-bounded symmetric polynomials.

Remarks:

- Doty and Walker (1992) mention the following generalization of Newton's identities, which they attribute to Macdonald:*

$$h_k^{(b)}(z_1,\dots,z_n) = \det\begin{pmatrix} p_1^{(b)} & p_2^{(b)} & \cdots & \cdots & p_k^{(b)} \\ -1 & p_1^{(b)} & p_2^{(b)} & & \vdots \\ & -2 & p_1^{(b)} & p_2^{(b)} & \vdots \\ & & \ddots & \ddots & p_2^{(b)} \\ & & & -(k-1) & p_1^{(b)} \end{pmatrix}$$

where

$$p_m^{(b)} = \begin{cases} (1-b)(z_1^m + \cdots + z_n^m) & b \mid m, \\ z_1^m + \cdots + z_n^m & b \nmid m. \end{cases}$$

* They did not express it as a determinant.

Remarks:

- This has an interesting consequence when $z_1 = \cdots = z_n = 1$:

$$\binom{n}{k}^{(b)} = \sum_{\lambda \vdash k} \frac{1}{z_\lambda}(1-b)^{l_b(\lambda)} n^{l(\lambda)},$$

where the sum is over $(\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ with $\sum_i \lambda_i = k$, and

$$l(\lambda) = \#\{i : \lambda_i \neq 0\},$$
$$l_b(\lambda) = \#\{i : b|\lambda_i\},$$
$$m_j = \#\{j : m_j = i\},$$
$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!.$$

Remarks:

- In a recent paper (*Lattice points and q-Catalan*, 2024) I proved that

$$\frac{1}{[n+1]_q} \sum_{k=\ell}^{m} q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^{(n+1)} \in \mathbb{Z}[q]$$

  whenever $\gcd(n+1, \ell-1) = \gcd(n+1, m) = 1$, and I conjectured that the coefficients are positive. I called these *q-Catalan germs*.

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- I don't know how this generalizes to $b \neq n+1$.

3. A Bit of Galois Theory

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Our main theorem will compute

$$\begin{bmatrix} n \\ k \end{bmatrix}_q^{(b)} \qquad \text{when } q \to \text{roots of unity.}$$

Before stating the theorem, it is worthwhile to mention a very general phenomenon, which follows from some basic Galois theory. This phenomenon is surely well known but I have not seen it written down.

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Observation

Let $f(z_1, \ldots, z_n) \in \mathbb{Z}[z_1, \ldots, z_n]$ be symmetric polynomial in n variables and let ω be a primitive dth root of unity for some d.

(a) If $d|n$ then $f(1, \omega, \ldots, \omega^{n-1}) = f(\omega, \ldots, \omega^n)$ is an integer.*

(b) If $d|(n-1)$ then $f(1, \omega, \ldots, \omega^{n-1})$ is an integer.

(c) If $d|(n+1)$ then $f(\omega, \ldots, \omega^n)$ is an integer.

* If $\deg(f) = k$ and $d \nmid k$ then this integer is zero.

Proof Sketch: (1) Let ω be a primitive dth root of unity and consider the field extension $\mathbb{Q}(\omega)/\mathbb{Q}$. The Galois group is

$$\mathrm{Gal}(\mathbb{Q}(\omega)/\mathbb{Q}) = \{\varphi_r : \gcd(r, d) = 1\},$$

where $\varphi_r : \mathbb{Q}(\omega) \to \mathbb{Q}(\omega)$ is defined by $\varphi_r(\omega) := \omega^r$. If $\alpha \in \mathbb{Z}[\omega]$ satisfies $\varphi_r(\alpha) = \alpha$ for all $\gcd(r, d) = 1$ then Galois theory tells us that $\alpha \in \mathbb{Z}$.

(2) Consider the sequence $\boldsymbol{\omega} := (\omega, \ldots, \omega^{d-1})$. If $\gcd(r, d) = 1$ then φ_r permutes the sequence $\boldsymbol{\omega}$, hence it permutes sequences of the following four types:

$$(1, \boldsymbol{\omega}, \ldots, \boldsymbol{\omega}, 1),$$
$$(\boldsymbol{\omega}, 1, \ldots, \boldsymbol{\omega}, 1),$$
$$(1, \boldsymbol{\omega}, 1, \boldsymbol{\omega}, \ldots, \boldsymbol{\omega}, 1),$$
$$(\boldsymbol{\omega}, 1, \boldsymbol{\omega}, 1, \ldots, 1, \boldsymbol{\omega}).$$

□

Corollary

Let ω be a primitive dth root of unity.

(a) If $d|n$ then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) = h_k^{(b)}(\omega, \ldots, \omega^n) \in \mathbb{Z}.$$

(b) If $d|(n-1)$ then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = h_k^{(b)}(1, \omega, \ldots, \omega^{n-1}) \in \mathbb{Z}.$$

(c) If $d|(n+1)$ then

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_{\omega}^{(b)} = h_k^{(b)}(\omega, \ldots, \omega^n) \in \mathbb{Z}.$$

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Our main theorem will compute these integers.

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$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = (1+t+\cdots+t^{b-1})(1+t^d+\cdots+(t^d)^{b-1})^{(n-1)/d}, \text{ i.e.,}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \sum_\ell \binom{(n-1)/d}{(k-\ell)/d}^{(b)} \geq 0.$$

4. The Main Theorem and Cyclic Sieving

Main Theorem (in three parts)

Let ω be a primitive dth root of unity with $\gcd(b, d) = 1$.

(c) If $d | (n + 1)$ then

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \frac{(1 + t^d + \cdots + (t^d)^{b-1})^{(n+1)/d}}{1 + t + \cdots + t^{b-1}} \in \mathbb{Z}[t].$$

These coefficients are sometimes negative and are more difficult to describe. We will give an explicit formula below in terms of the Frobenius Coin Problem.

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Remark: My paper also gives explicit generating functions for (a),(b),(c) when $\gcd(b,d)\neq 1$, which are more complicated.

4. The Main Theorem and Cyclic Sieving

Parts (a) and (b) have a nice combinatorial interpretation, in terms of cyclic sieving (Reiner-Stanton-White, 2004). Again, consider the set of points in a diagonal slice of the integer box $\{0, 1, \ldots, b-1\}^n$:

$$X = \{(x_1, \ldots, x_n) \in \{0, 1, \ldots, b-1\}^n : x_1 + x_2 + \cdots + x_n = k\}.$$

This set is closed under permutations. Consider the following two permutations:

$$\rho \cdot (x_1, \ldots, x_n) := (x_2, \ldots, x_n, x_1),$$
$$\tau \cdot (x_1, \ldots, x_n) := (x_2, \ldots, x_{n-1}, x_1, x_n).$$

Note that $\langle \rho \rangle \cong \mathbb{Z}/n\mathbb{Z}$ and $\langle \tau \rangle \cong \mathbb{Z}/(n-1)\mathbb{Z}$. Recall that we can identify X with k-subsets and k-multisubsets of $\{1, \ldots, n\}$ when $b = 2$ and $b = \infty$.

Corollary of Main Theorem

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I find the condition $\gcd(b, d) = 1$ surprising!

Remarks:

- This result generalizes the prototypical examples of cyclic sieving (Theorem 1.1 in RSW) for k-subsets (when $b = 2$) and k-multisubsets (when $b = \infty$).
- I find it surprising that it was not already known to the experts.
- Our Main Theorem (a),(b) generalizes Prop 4.2 in RSW, which appears there as a random collection of identities.
- Main Theorem (c) has no analogue in RSW.
- It may be interesting to look at the integers $f(\omega, \ldots, \omega^n) \in \mathbb{Z}$ when $d|(n+1)$ for other classes of symmetric polynomials.

5. The Frobenius Coin Problem

Let ω be a primitive dth root of unity with $d|(n+1)$ and $\gcd(b,d)=1$. Recall that

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{(1+t^d+\cdots+(t^d)^{b-1})^{(n+1)/d}}{1+t+\cdots+t^{b-1}} \in \mathbb{Z}[t].$$

The integers $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ are not directly related to cyclic sieving.

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Using the notation $[n]_t = 1+t+\cdots+t^{n-1}$ we can write this as

$$\sum_k \omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} t^k = \frac{[b]_{t^d}}{[b]_t} [b]_{t^d}^{(n+1)/d-1}.$$

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We want to study the coefficients of the polynomial

$$\frac{[b]_{t^d}}{[b]_t} \in \mathbb{Z}[t].$$

It turns out these coefficients are related to the Frobenius Coin Problem.

5. The Frobenius Coin Problem

Given integers $\gcd(b, d) = 1$, consider the function $\nu_{b,d} : \mathbb{N} \to \mathbb{N}$,

$$\nu_{b,d}(n) := \#\{(k, \ell) \in \mathbb{N}^2 : bk + d\ell = n\}.$$

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The set of non-representable numbers is finite, called the Sylvester set:

$$S_{b,d} = \{n \in \mathbb{N} : \nu_{b,d}(n) = 0\}.$$

For example, $S_{3,5} = \{1, 2, 4, 7\}$. Sylvester (1882) proved that

$$\#S_{b,d} = (b-1)(d-1)/2 \quad \text{and} \quad \max(S_{b,d}) = bd - b - d.$$

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$$\#S_{b,d} = (b-1)(d-1)/2 \quad \text{and} \quad \max(S_{b,d}) = bd - b - d.$$

Let us define the Sylvester polynomial

$$S_{b,d}(t) := \sum_{s \in S_{b,d}} t^s.$$

For example, $S_{3,5}(t) = t + t^2 + t^4 + t^7$.

Brown and Shiue (1993) attribute the following result to Ozluk.

Theorem (Ozluk)

If $\gcd(b, d) = 1$ then we have $[b]_{t^d}/[b]_t = 1 + (t-1)S_{b,d}(t)$, i.e.,

$$S_{b,d}(t) = \frac{t^{bd} - 1}{(1 - t^b)(1 - t^d)} + \frac{1}{1 - t}.$$

5. The Frobenius Coin Problem

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Corollary

If ω is a primitive dth root of unity with $d|(n+1)$, it follows that

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} = \binom{(n+1)/d-1}{k/d}^{(b)} + \sum_{s\in S_{b,d}} \binom{(n+1)/d-1}{(k-1-s)/d}^{(b)} - \sum_{s\in S_{b,d}} \binom{(n+1)/d-1}{(k-s)/d}^{(b)}.$$

It is not clear from this formula when $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)}$ is positive or negative.

5. The Frobenius Coin Problem

Here is a cute formula, which allows us to be much more precise.

Theorem

Let $\gcd(b, d) = 1$. For any $r \in \mathbb{N}$, let $0 \le \beta_r < b$ and $0 \le \delta_r < d$ satisfy

$$\beta_r \equiv rd^{-1} \bmod b \quad \text{and } \delta_r \equiv rb^{-1} \bmod d.$$

Then

$$\frac{[b]_{t^d}}{[b]_t} = \frac{[d]_{t^b}}{[d]_t} = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b - \beta_1]_{t^d}[d - \delta_1]_{t^b}.$$

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Corollary

Let $\gcd(b, d) = 1$. If ω is a primitive dth root of unity and $d|(n+1)$ then

$$\omega^k \begin{bmatrix} n \\ k \end{bmatrix}_\omega^{(b)} \text{ is } \begin{cases} \geq 0 & \text{when } \delta_k < \delta_1, \\ \leq 0 & \text{when } \delta_k \geq \delta_1. \end{cases}$$

5. The Frobenius Coin Problem

I really like this theorem because it has a geometric interpretation.

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14	19	24	.	.	.	.	.	.	.	.
7	12	17	.	.	.	.	.	.	.	.
0	5	10	15	20	25	30	35	.	.	.
		3	8	13	18	23	28	.	.	.
			1	6	11	16	21	.	.	.
					4	9	14	19	24	.
						2	7	12	17	.
							0	5	10	.

Example: Let $(b, d) = (7, 5)$. Draw an infinite array starting at 0, adding 5 for each right step and subtracting 7 for each down step.

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Example: Let $(b, d) = (7, 5)$. Draw an infinite array starting at 0, adding 5 for each right step and subtracting 7 for each down step.

The Sylvester set forms a triangle:

$$S_{7,5} = \{1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23\}.$$

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In this case we have $(\beta_1, \delta_1) = (3, 3)$, which tells us that the label 1 occurs in position $(\beta_1, \delta_1 - d) = (3, -2)$.

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The cute formula describes two rectangles with bottom corners at 0 and 1.

$$[b]_{t^d}/[b]_t = [\beta_1]_{t^d}[\delta_1]_{t^b} - t[b-\beta_1]_{t^d}[d-\delta_1]_{t^b}$$

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\begin{aligned}
[7]_{t^5}/[7]_t &= [3]_{t^5}[3]_{t^7} - t[4]_{t^5}[2]_{t^7} \\
&= 1 + t^5 + t^7 + t^{10} + t^{12} + t^{14} + t^{17} + t^{19} + t^{24} \\
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\end{aligned}
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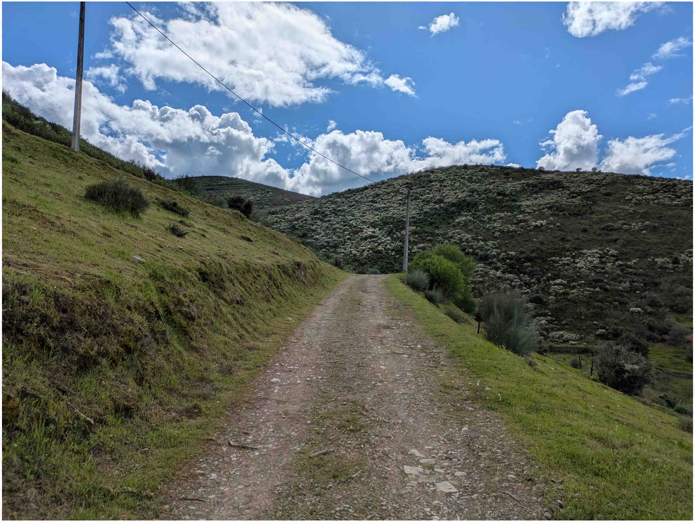
\end{aligned}
$$

And this leads to a precise description of $\omega^k \begin{bmatrix} n \\ k \end{bmatrix}^{(7)}_{\omega}$ when $\omega^5 = 1$.

Obrigado!



Thanks to DeepSeek for suggesting the azulejos background image.