Rational Noncrossing Partitions and Associahedra

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“Recent” Perspectives on Non-crossing Partitions, etc.
ESI, Vienna, February 2025

Public Service Announcement

Beamer places strange navigation symbols in the bottom right corner and no one even knows what they're for. Here's how to get rid of them:

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Plan

This talk is based on joint work with Loehr, Rhoades, Warrington and Williams from approximately 12 years ago. Is that "recent"?

1. Given any $x \in \mathbb{Q}$ define the Catalan number $\mathrm{Cat}(x) \in \mathbb{Z}$.
2. Given any $x \in \mathbb{Q}$ with $x > 0$ define noncrossing partitions $\mathrm{NC}(x)$.
3. Given any $x \in \mathbb{Q}$ with $x > 0$ define the associahedron $\mathrm{Ass}(x)$.
4. Given any $x \in \mathbb{Q}$ with $x > 0$ define parking functions $\mathrm{PF}(x)$.

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What is a Catalan Number?

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Some Strange Ideas

Given $x \in \mathbb{Q} \setminus \left\{-1, -\frac{1}{2}, 0\right\}$ there exist unique coprime integers $a, b \in \mathbb{Z}$ with $0 < a < |b|$ or $0 < b < |a|$ such that

$$x = \frac{a}{b-a}.$$

(Note that $0 < x \iff 0 < a < b$.) We will always identify $x \leftrightarrow (a, b)$.

Examples: Given $1 \leq n \in \mathbb{N}$ we have

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Examples: Given $1 \leq n \in \mathbb{N}$ we have

$$x = n \leftrightarrow (n, n+1)$$

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Examples: Given $1 \leq n \in \mathbb{N}$ we have

$$x = \frac{1}{n} \leftrightarrow (1, n+1)$$

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Examples: Given $1 \leq n \in \mathbb{N}$ we have

$$x = -n \leftrightarrow (n, n-1) \quad \text{(need } n \geq 2\text{)}$$

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Examples: Given $1 \leq n \in \mathbb{N}$ we have

$$x = -\frac{1}{n} \leftrightarrow (1, -n+1) \quad \text{(need } n \geq 3\text{)}$$

What is a Catalan Number?

Definition

For each $x \in \mathbb{Q} \setminus \left\{-1, -\frac{1}{2}, 0\right\}$ we define the Catalan number:

$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) := \frac{1}{a+b}\binom{a+b}{a, b}.$$

Claim: This is an integer. (Proof postponed.)

Example:

$$\mathrm{Cat}\left(\frac{5}{3}\right) = \mathrm{Cat}\left(\frac{5}{8-5}\right) = \mathrm{Cat}(5, 8) = \frac{1}{13}\binom{13}{5, 8} = 99.$$

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Classical Cases

When $b = 1 \bmod a$ we have ...

- *Eugène Charles Catalan (1814-1894)*

  $(a, b) = (n, n+1)$ gives the **good old Catalan number**:

$$\mathrm{Cat}(n) = \mathrm{Cat}\left(\frac{n}{(n+1)-n}\right) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

- *Nicolaus Fuss (1755-1826)*

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$$\mathrm{Cat}(x) = \mathrm{Cat}(a, b) = \mathrm{Cat}(b, a) = \mathrm{Cat}(-x - 1).$$

This implies that for $x \in \mathbb{Q} \setminus \left\{-1, -\frac{1}{2}, 0\right\}$ we have

$$\mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

We will call this the derived Catalan number:

$$\mathrm{Cat}'(x) := \mathrm{Cat}\left(\frac{1}{x-1}\right) = \mathrm{Cat}\left(\frac{x}{1-x}\right).$$

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Given $0 < x$ (i.e. $0 < a < b$) note that we have

$$\mathrm{Cat}'(1/x) = \mathrm{Cat}\left(\frac{1}{(1/x) - 1}\right) = \mathrm{Cat}\left(\frac{x}{1 - x}\right) = \mathrm{Cat}'(x).$$

We call this rational duality:

$$\mathrm{Cat}'(x) = \mathrm{Cat}'(1/x).$$

In terms of coprime $0 < a < b$ this translates to

$$\mathrm{Cat}'(a, b) = \mathrm{Cat}'(b - a, b).$$

This will appear later as **Alexander duality** of rational associahedra.

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Euclidean Algorithm

Observation

Given $0 < a < b$ coprime, we observe that

$$\mathrm{Cat}'(a,b) = \frac{1}{b}\binom{b}{a} = \begin{cases} \mathrm{Cat}(a, b-a) & \text{for } a < (b-a) \\ \mathrm{Cat}(b-a, a) & \text{for } (b-a) < a \end{cases}$$

This allows us to define a sequence

$$\mathrm{Cat}(x) \mapsto \mathrm{Cat}'(x) \mapsto \mathrm{Cat}''(x) \mapsto \cdots$$

which is a Categorification of the Euclidean algorithm.

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Euclidean Algorithm

Example: $x = 5/3$ and $(a, b) = (5, 8)$

Subtract the smaller from the larger:

$$\begin{aligned}
\mathrm{Cat}(5,8) &= 99,\\
\mathrm{Cat}'(5,8) &= \mathrm{Cat}(3,5) = 7,\\
\mathrm{Cat}''(5,8) &= \mathrm{Cat}'(3,5) = \mathrm{Cat}(2,3) = 2,\\
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A Strange Idea

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Suggestion

Extend the function $\mathrm{Cat} : \mathbb{Q} \to \mathbb{N}$ analytically to the upper half plane.

The Prototype: Rational Dyck Paths

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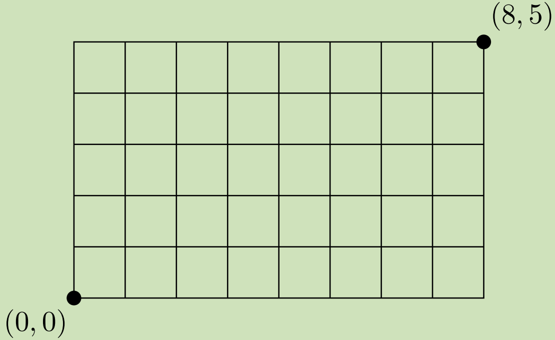
- Consider the "Dyck paths" in an $a \times b$ rectangle.

Example $(a, b) = (5, 8)$



The Prototype: Rational Dyck Paths

- Again let $0 < x = a/(b-a)$ with $0 < a < b$ coprime.

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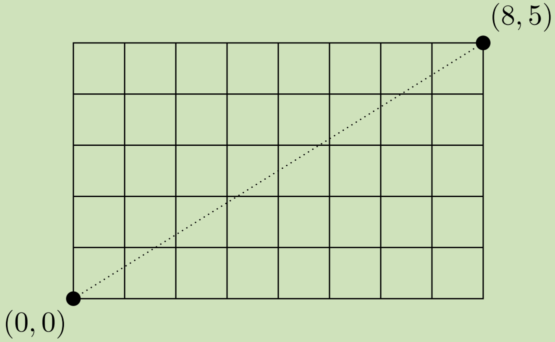

The Prototype: Rational Dyck Paths

- Let $\mathcal{D}(x) = \mathcal{D}(a, b)$ denote the set of Dyck paths.

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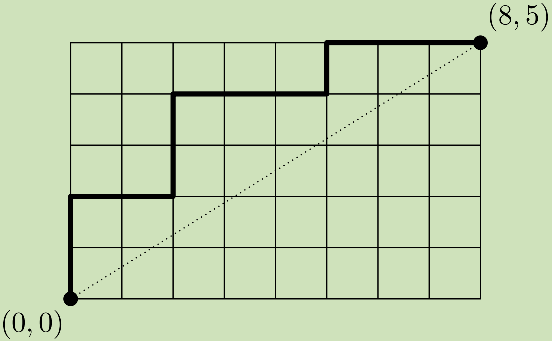

The Prototype: Rational Dyck Paths

Theorem (Grossman 1950, Bizley 1954)

*For a, b **coprime**, the number of Dyck paths is the Catalan number:*

$$|\mathcal{D}(x)| = \mathrm{Cat}(x) = \frac{1}{a+b}\binom{a+b}{a,b}.$$

- *Claimed by Grossman (1950), "Fun with lattice points, part 22".*
- *Proved by Bizley (1954), in Journal of the Institute of Actuaries.*
- *Proof: Break $\binom{a+b}{a,b}$ lattice paths into cyclic orbits of size $a+b$. Each orbit contains a unique Dyck path.*

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The Prototype: Rational Dyck Paths

Theorem (with N. Loehr and G. Warrington)

- *The number of Dyck paths with k vertical runs equals*

$$\mathrm{Nar}(x;k) := \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}.$$

Call these the **Narayana numbers***.*

- *And the number with r_j vertical runs of length j equals*

$$\mathrm{Krew}(x;\mathbf{r}) := \frac{1}{b}\binom{b}{r_0, r_1, \ldots, r_a} = \frac{(b-1)!}{r_0! r_1! \cdots r_a!}.$$

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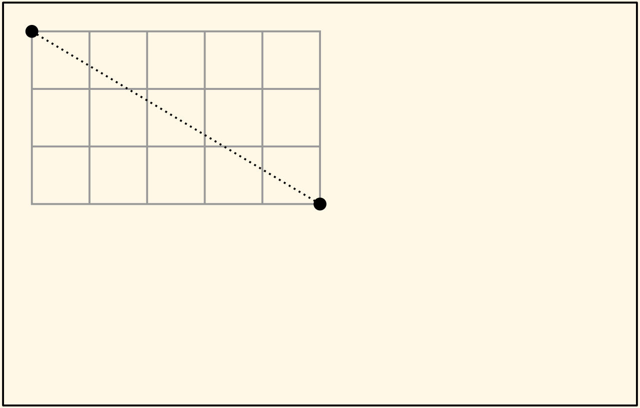
Bizley's Proof

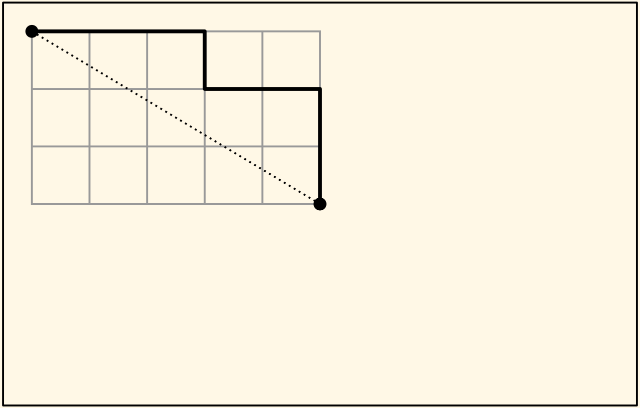
I will present Bizley's proof of the theorem.

For example, suppose that $(a, b) = (3, 5)$.



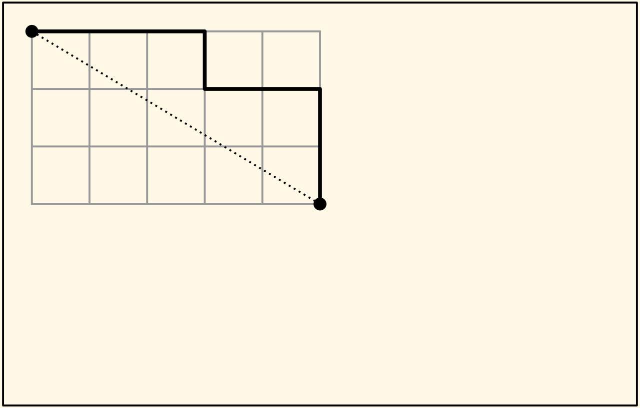
Bizley's Proof

There are a total of $\binom{a+b}{a,b}$ lattice paths from $(0,0)$ to $(b,-a)$.



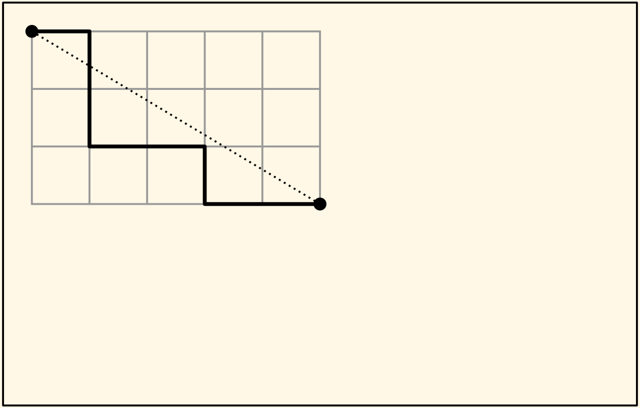
Bizley's Proof

Some of them are above the diagonal.



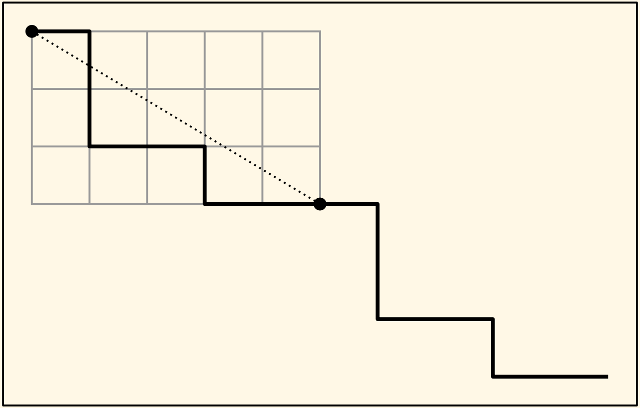
Bizley's Proof

. . . and some of them are not.



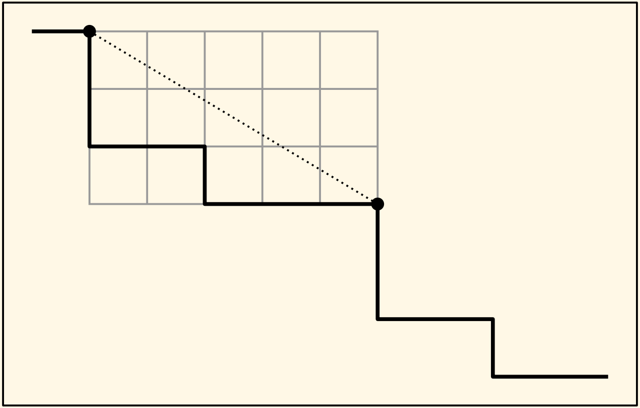
Bizley's Proof

If we double a given path . . .



Bizley's Proof

. . . then we can rotate it to create more paths.



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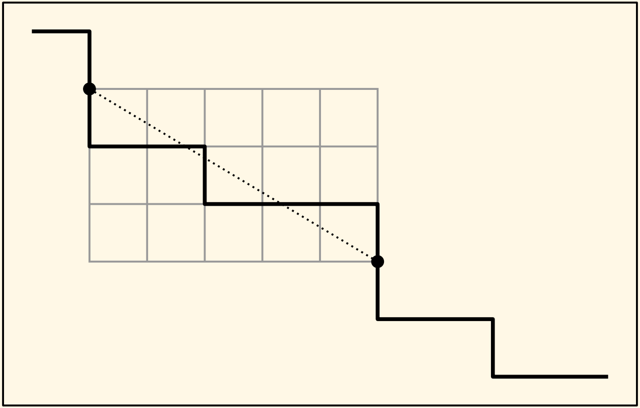

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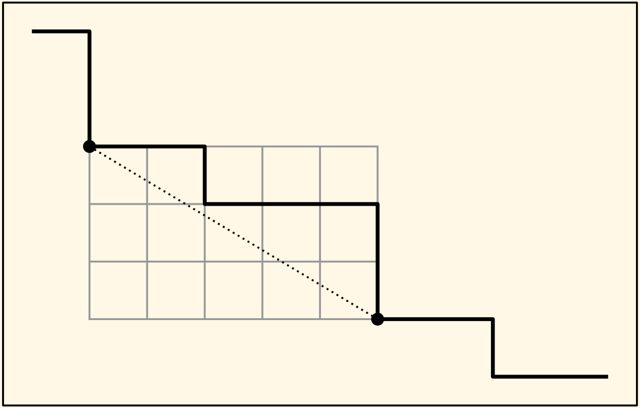

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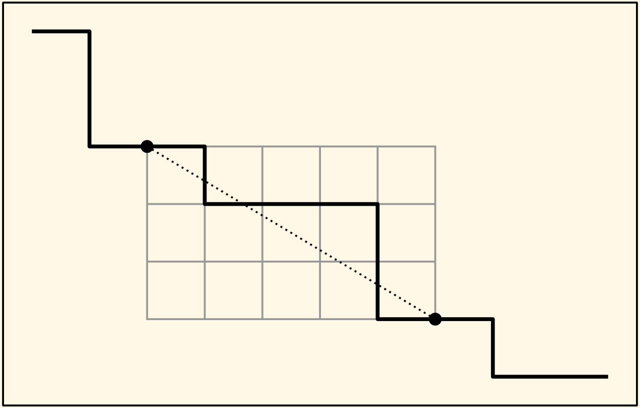

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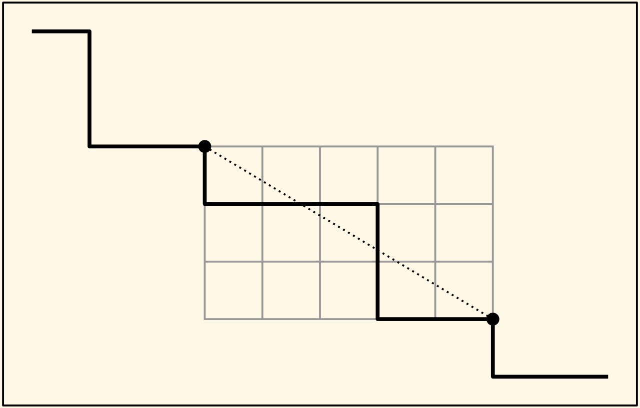

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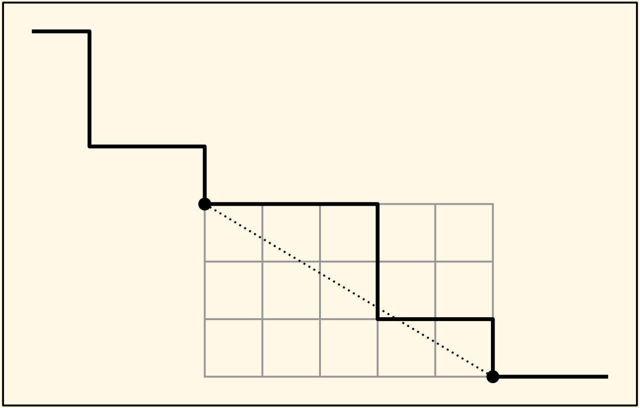

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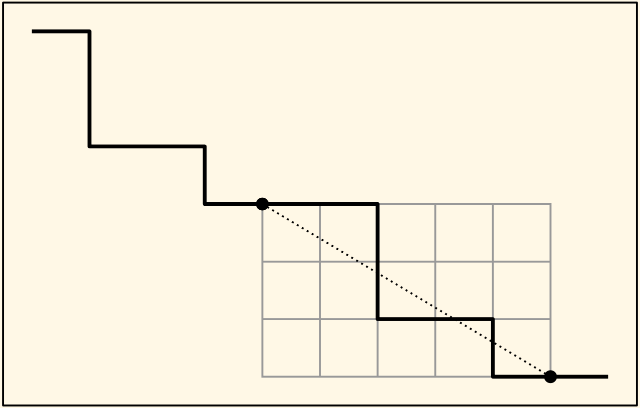

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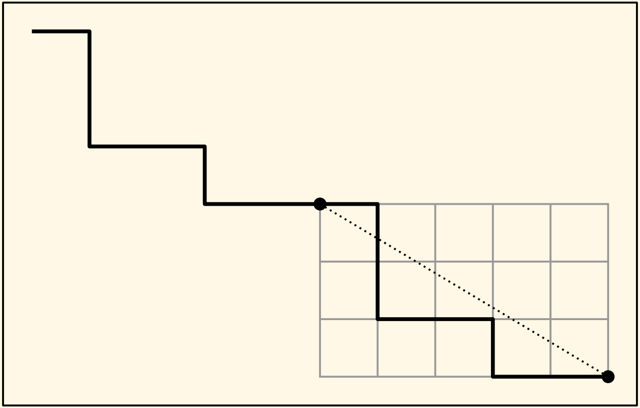
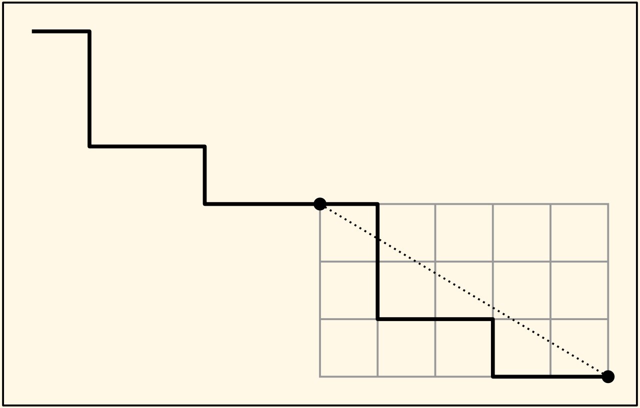


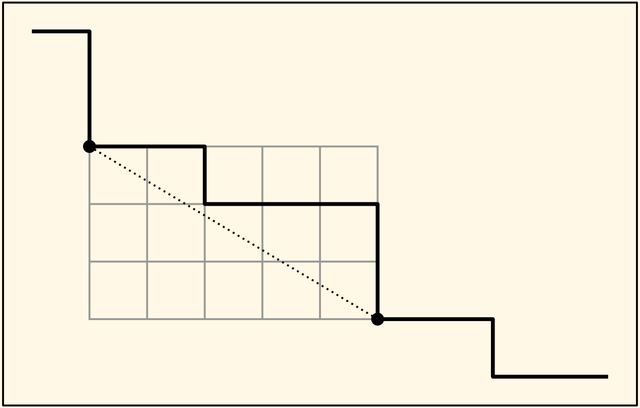
Bizley's Proof

Since $\gcd(a, b) = 1$, there are $a + b$ distinct rotations of each path.



Bizley's Proof

. . . and exactly one of them is above the diagonal.



Bizley's Proof

Thus we obtain a bijection

$$\text{(Dyck paths)} \quad \longleftrightarrow \quad \text{(rotation classes of paths)}$$

and it follows that

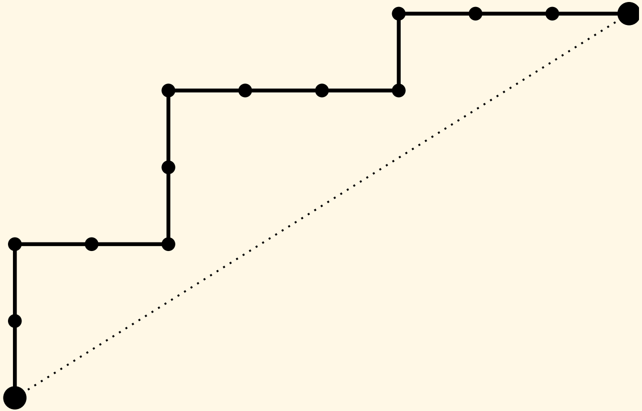
$$\#\text{(Dyck paths)} = \binom{a+b}{a, b} / (a+b).$$

This completes the proof of Bizley's Theorem. $\square$

Next: Rational NC Partitions

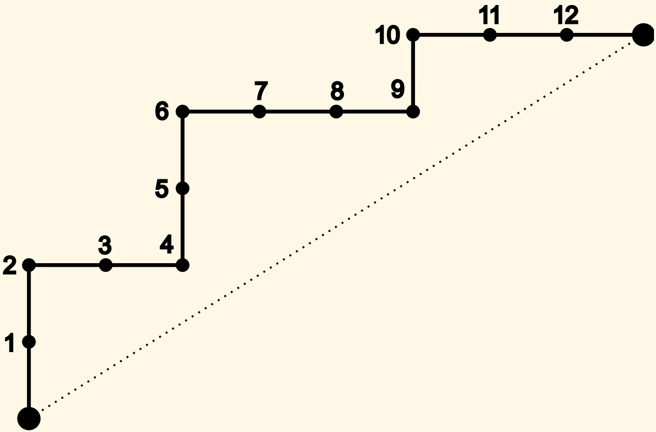
To **create** a noncrossing partition. . .

- Start with a Dyck path. Here $(a, b) = (5, 8)$.



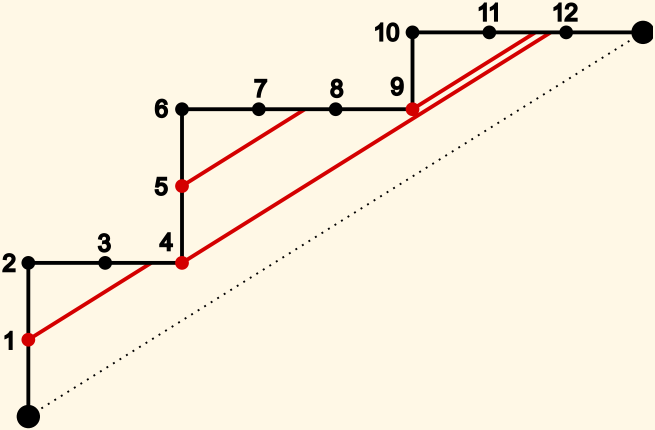
To **create** a noncrossing partition...

- Label the internal vertices by $\{1, 2, \ldots, a + b - 1\}$.



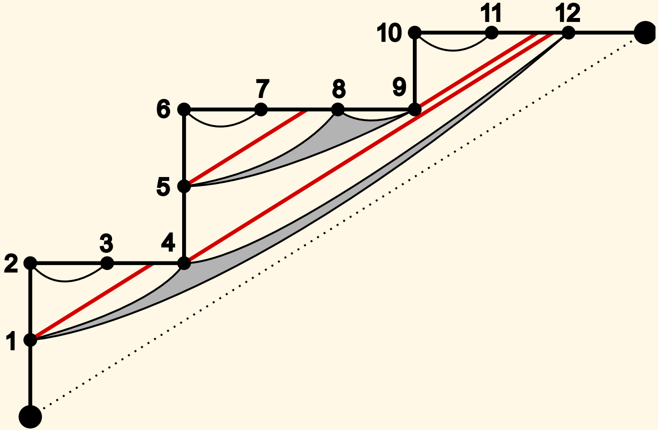
To **create** a noncrossing partition...

- Shoot lasers from the bottom left with slope a/b.



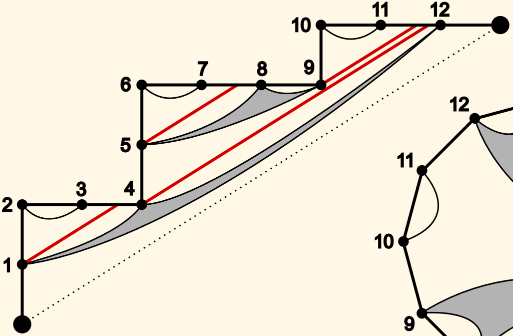
To **create** a noncrossing partition. . .

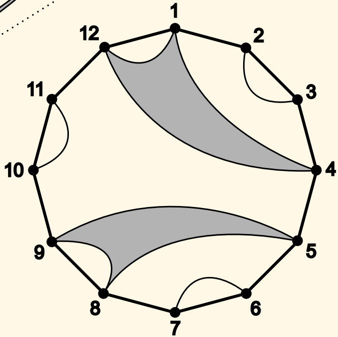
- Who can see each other?



To **create** a noncrossing partition...

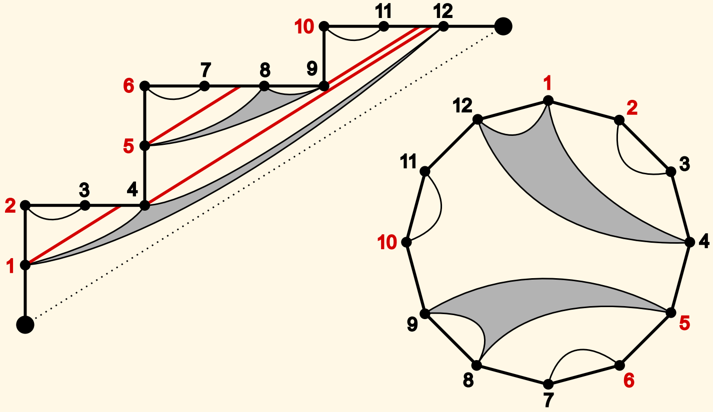
- There you go!





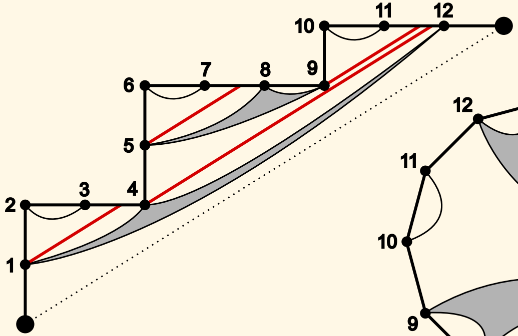
To **create** a noncrossing partition. . .

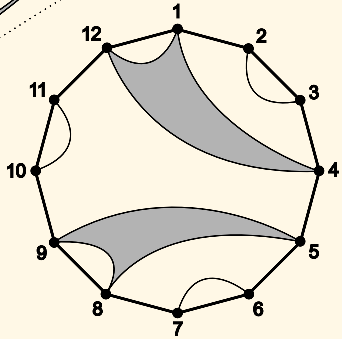
- We have created $\mathrm{Cat}(x) = \frac{1}{a}\binom{a+b}{a,b}$ different noncrossing partitions of the cycle $[a+b-1]$, and each of them has a blocks.



To **rotate** a noncrossing partition. . .

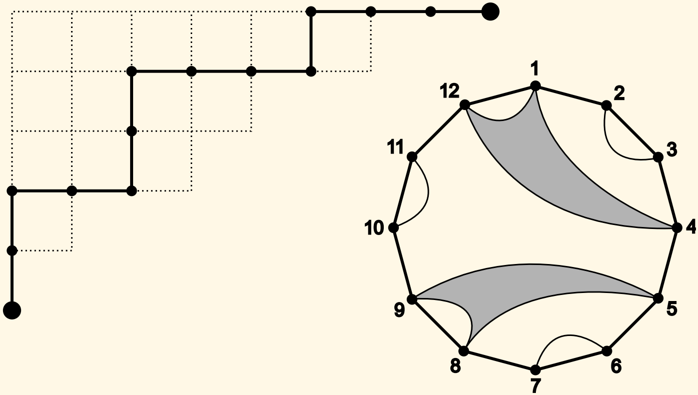
- Q: What does "rotation" of the partition correspond to?





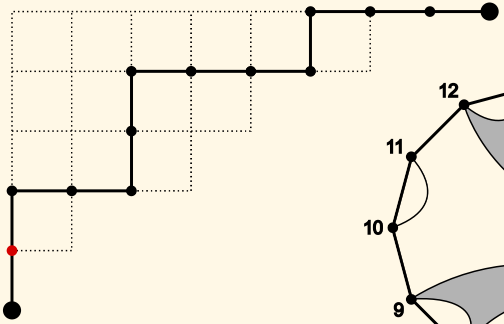
To **rotate** a noncrossing partition. . .

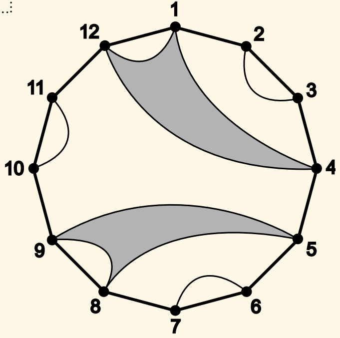
- A: Think of the path as a maximal chain in a poset.



To **rotate** a noncrossing partition. . .

- Perform "promotion" on the chain.





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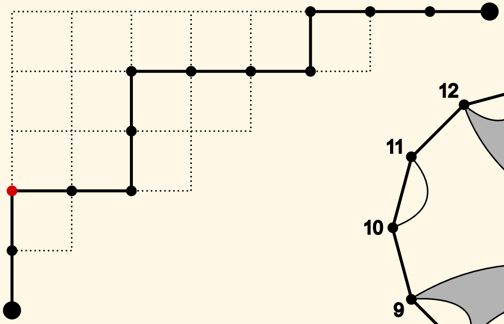

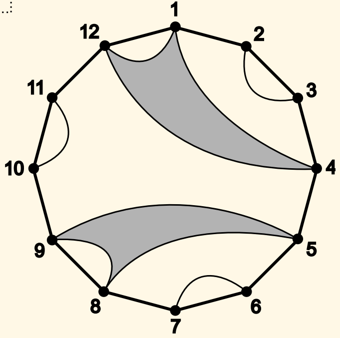

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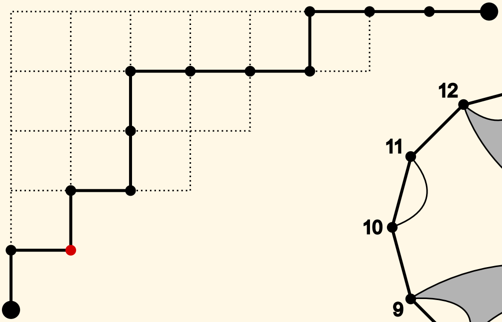

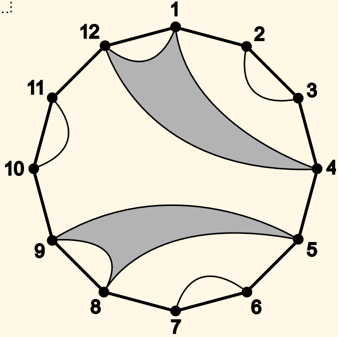

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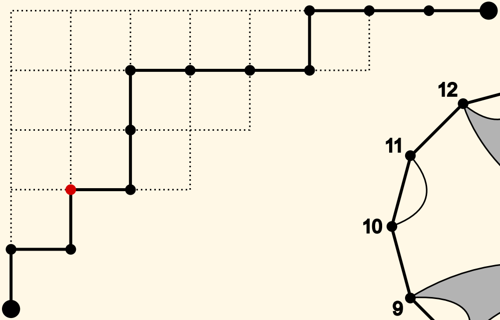

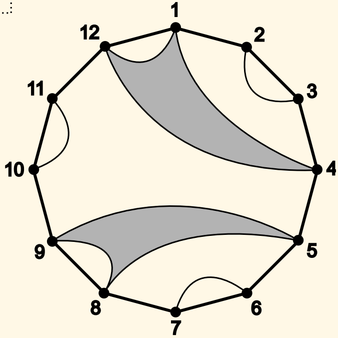

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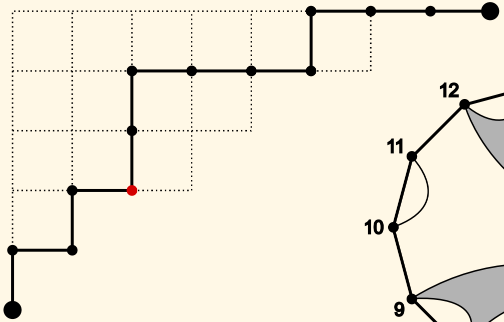

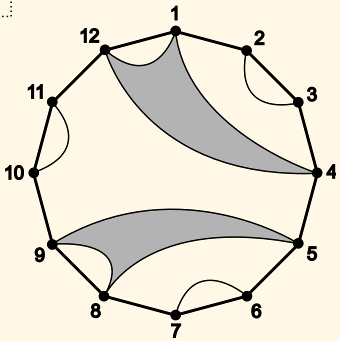

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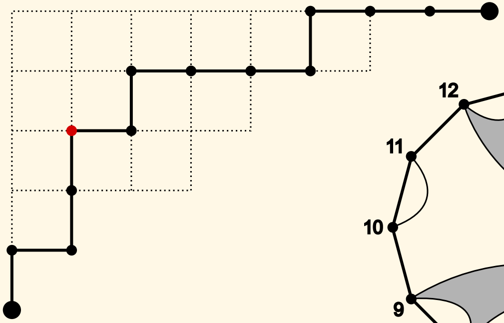

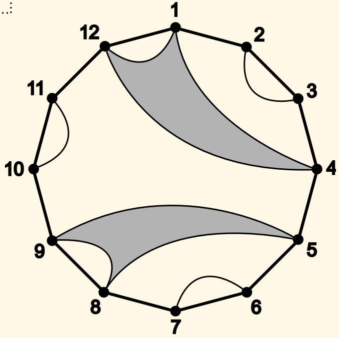

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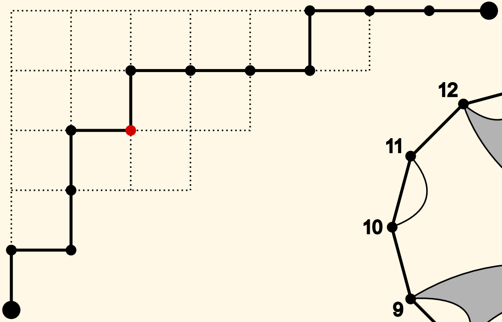

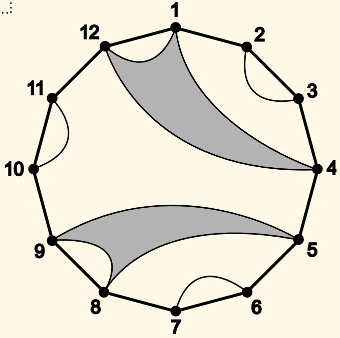

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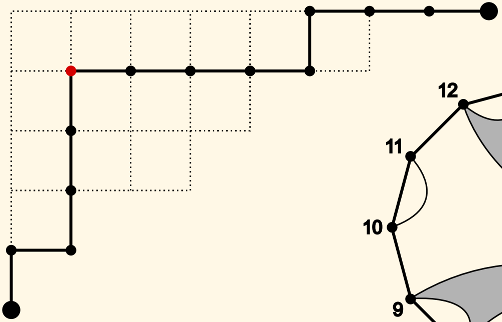

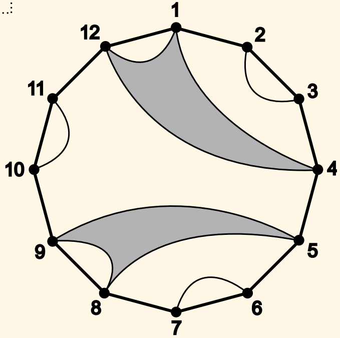

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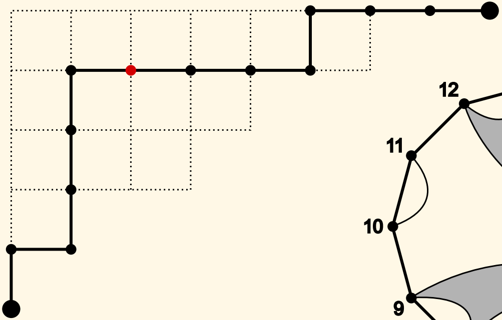

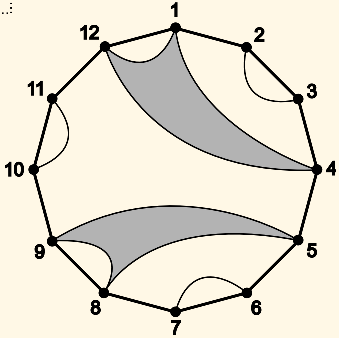

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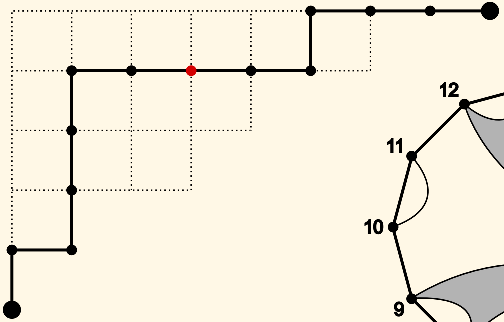

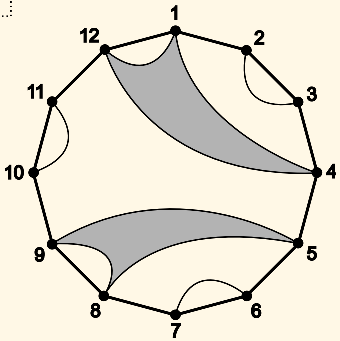

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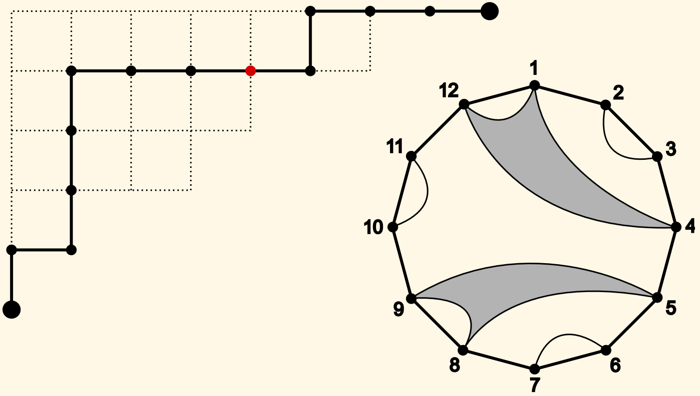

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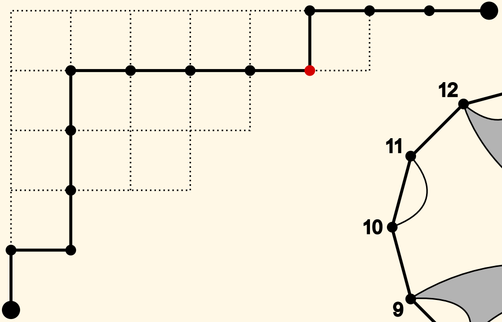

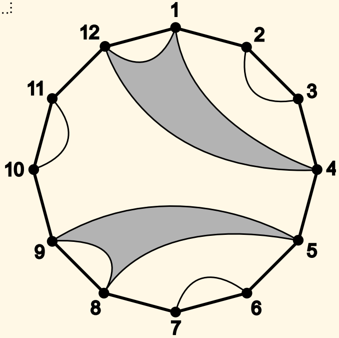

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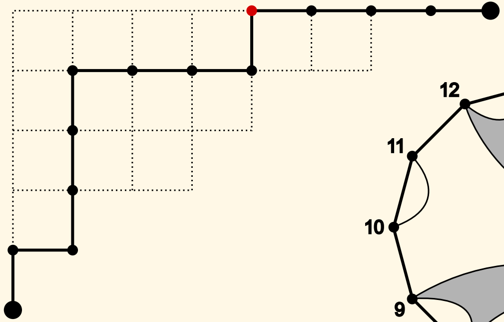

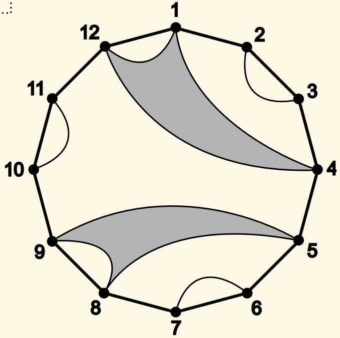

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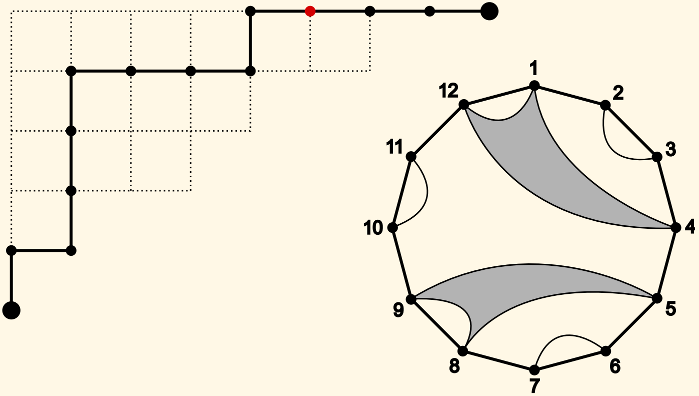

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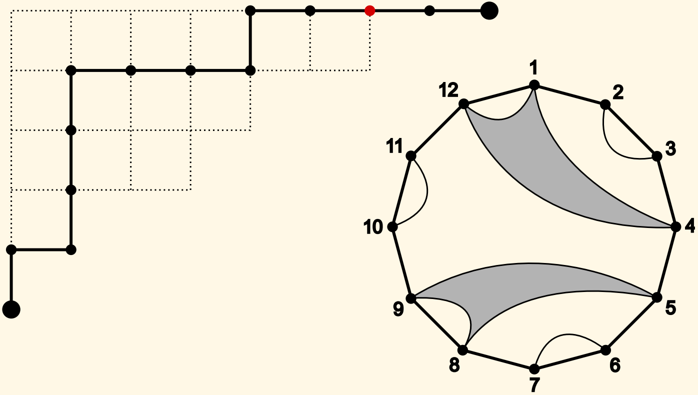

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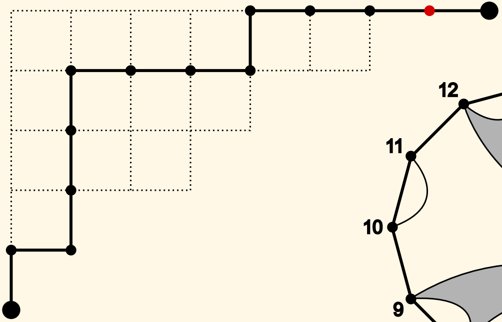

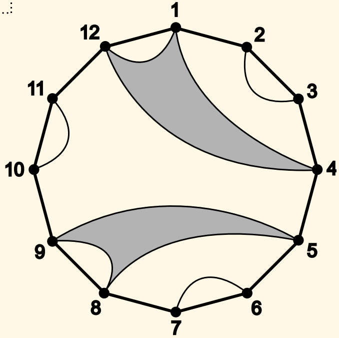

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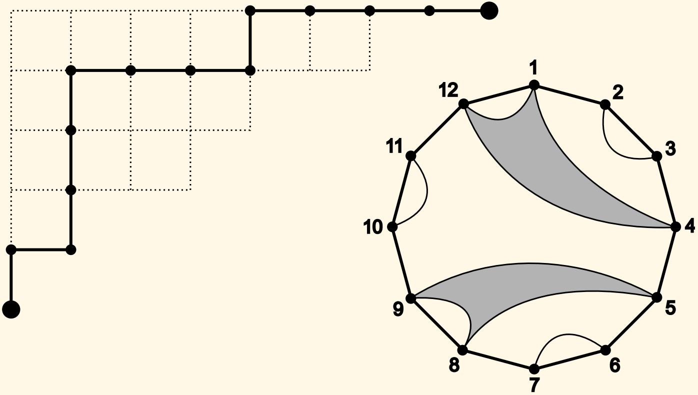
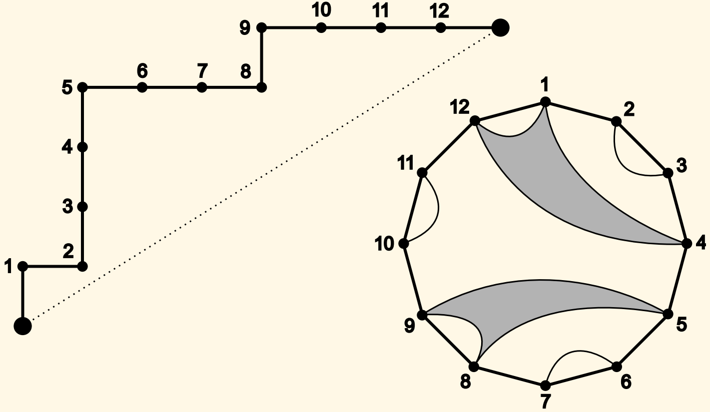


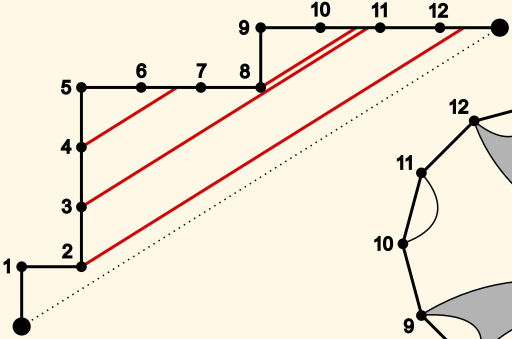
To **rotate** a noncrossing partition...

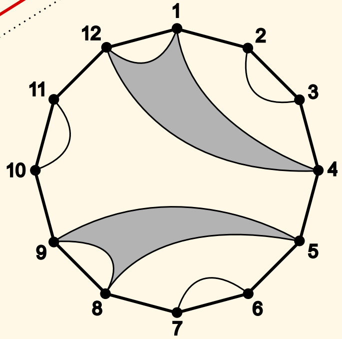
- Think of it as a path again.



To **rotate** a noncrossing partition...

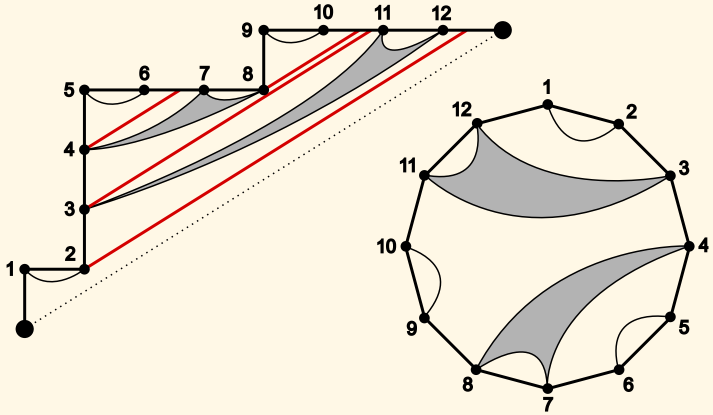
- Again the lasers.





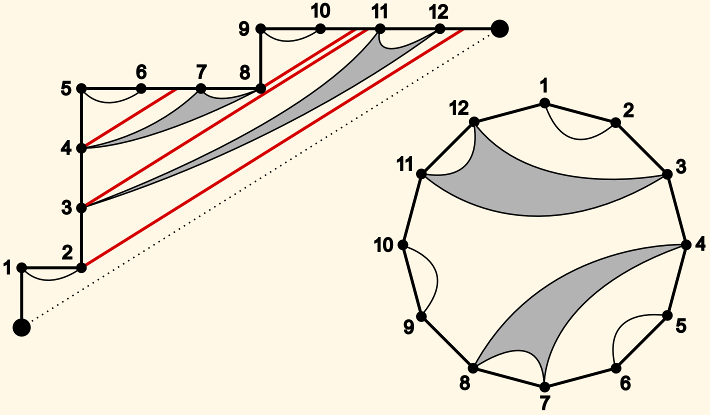
To **rotate** a noncrossing partition...

- And there you go!



To **rotate** a noncrossing partition...

- *Drew: mention the case* $(a, b) = (n, (k-1)n + 1)$.



NC Results

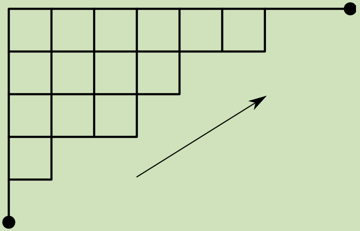
NC Results

Definition

For (a, b) coprime, consider the **triangle poset**

$$\mathcal{T}(a, b) := \{(x, y) \in \mathbb{Z}^2 : y \leq a,\ x \leq b,\ yb - xa \geq 0\}.$$

As you see here.



NC Results

Conjecture (with N. Williams)

- *Promotion on $\mathcal{T}(a,b)$ has order $a+b-1$.*
- *The number of chains invariant under promotion^d is the q-***Catalan number** *evaluated at an $(a+b-1)$th root of unity:*

$$\frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q \Bigg|_{q=e^{\frac{2\pi i d}{a+b-1}}}$$

Theorem (M. Bodnar and B. Rhoades)

The conjecture is true.

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Rational NC Partition Posets

Observation

Our rational NC partitions don't form a nice poset. Indeed, they each have the same number of blocks! (i.e., a)

Question

Can one define a **nice poset** of rational NC partitions?

Answer

Yes.

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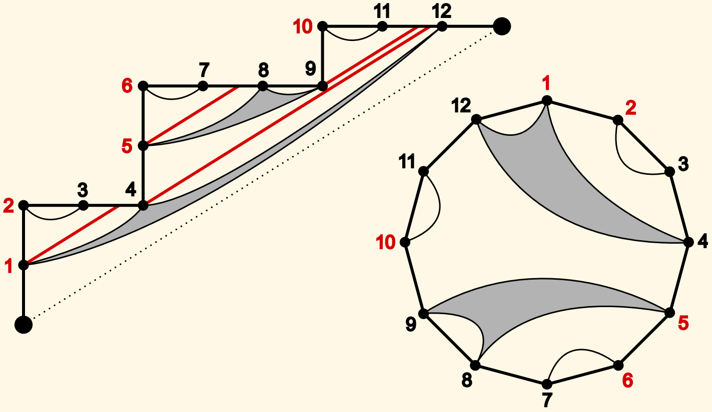
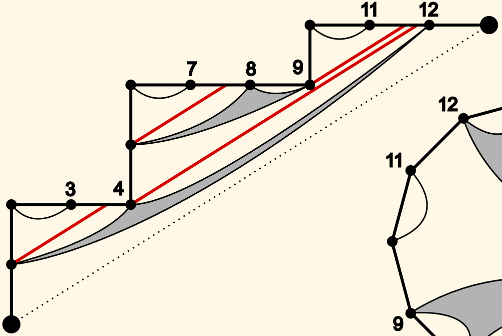
To **de-homogenize** a noncrossing partition...

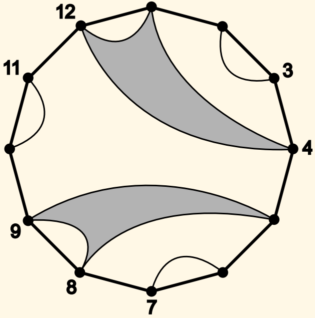
- Recall this.



To **de-homogenize** a noncrossing partition...

- Now we label only the horizontal steps.





To **de-homogenize** a noncrossing partition. . .

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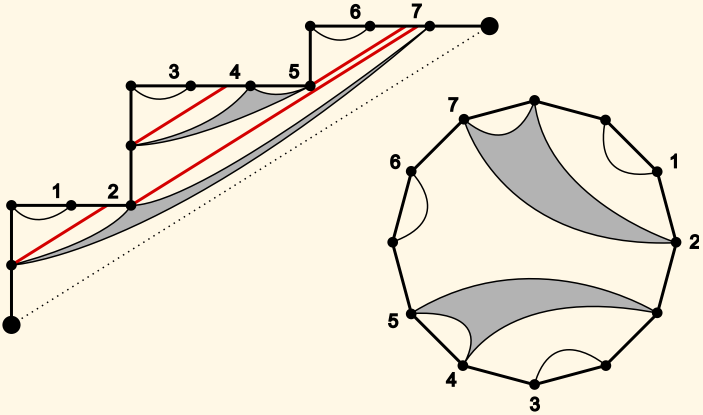
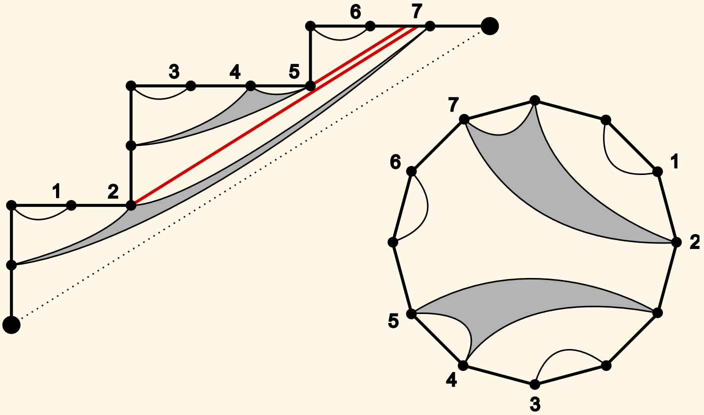


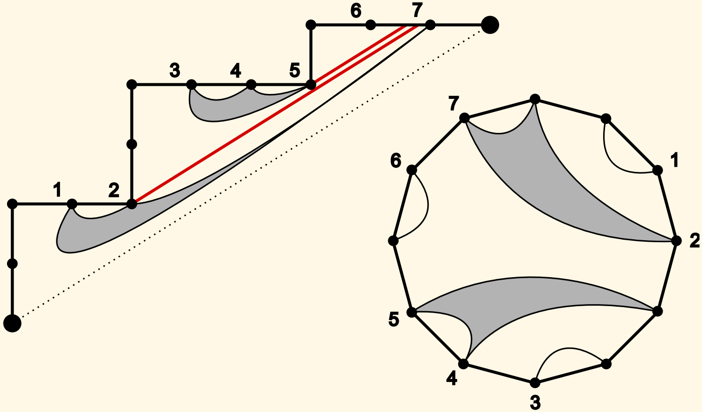
To **de-homogenize** a noncrossing partition...

- Now we shoot lasers only from the corners.



To **de-homogenize** a noncrossing partition. . .

- Now who can see each other?



To **de-homogenize** a noncrossing partition. . .

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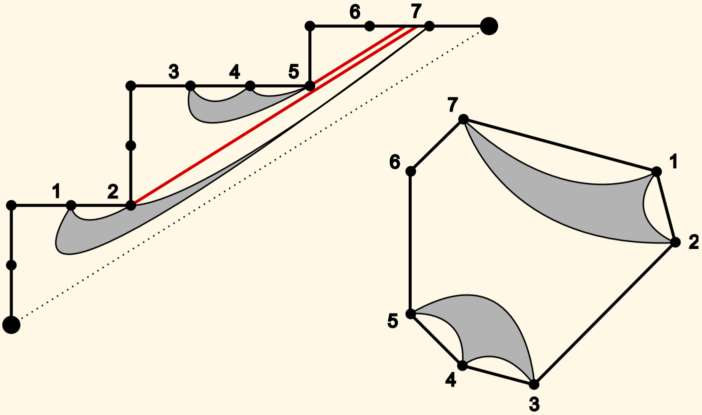

NC Poset Results

Definition

Let $\mathrm{NC}(x) = \mathrm{NC}(a, b)$ be the poset of non-homogeneous NC partitions.

Theorem (with B. Rhoades and N. Williams)

- $\mathrm{NC}(n, n+1) = \mathrm{NC}(n)$ is the **good old noncrossing partitions**.
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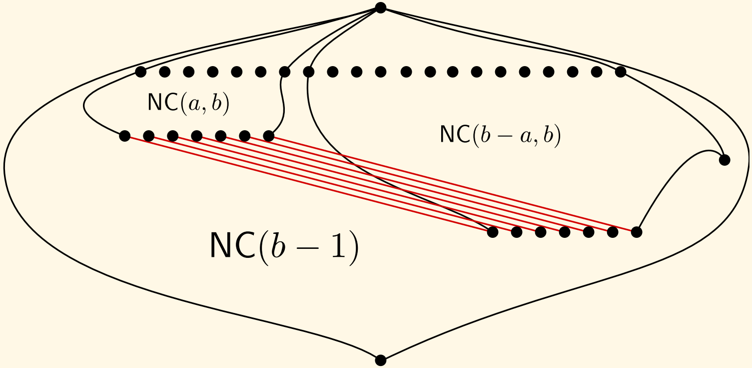
Rational Duality

- Note that $x \leftrightarrow 1/x$ is the same as $(a < b) \leftrightarrow (b - a < b)$.



Cyclic Sieving

Conjecture (with N. Williams)

- *The* $(b-1)$*-fold rotation action on* $\mathrm{NC}(b-1)$ *preserves the subposets* $\mathrm{NC}(a,b)$ *and* $\mathrm{NC}(b-a,b)$*.*
- *The number of elements of* $\mathrm{NC}(a,b)$ *invariant under rotation by d mod* $b-1$ *is the* q-**Catalan number evaluated** *at a* $(b-1)$*th root of unity:*

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Next: Rational Associahedra

The Classical Associahedron

Definition

Let $n \geq 0$ and consider a convex $(n+2)$-gon C. Let Ass(n) be the abstract simplicial complex with

- vertices = chords of C
- faces = noncrossing sets of chords of C
- maximal faces = triangulations of C

Theorem (Milnor, Haiman, C. Lee, etc.)

Ass(n) *is a polytope.*

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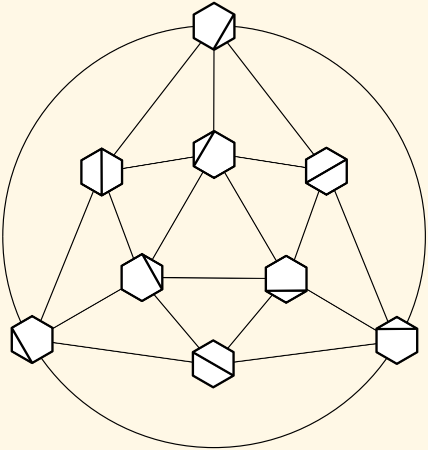
- Example: Here is Ass(4).



The Classical Associahedron

Theorem (Euler, 1751)

The f-vector and h-vector of Ass(n) *are given by the* **Kirkman numbers**

$$\mathrm{Kirk}(n;k) = \frac{1}{n}\binom{n}{k}\binom{n+k}{k-1}$$

and the **Narayana numbers**

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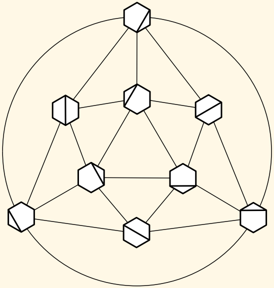
The Classical Associahedron

- Example: Here are the *f*-vector and *h*-vector of Ass(4) .

1
1 6
1 7 6
1 8 13 1
1 9 21 14



The Rational Associahedron

Idea

Given $0 < x = a/(b - a)$ with $0 < a < b$ coprime, we will define a simplicial complex

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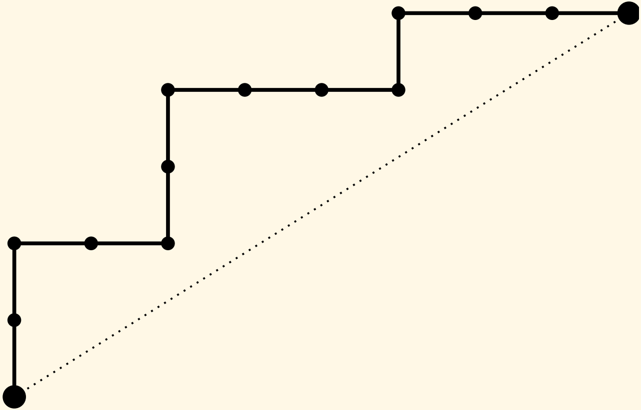
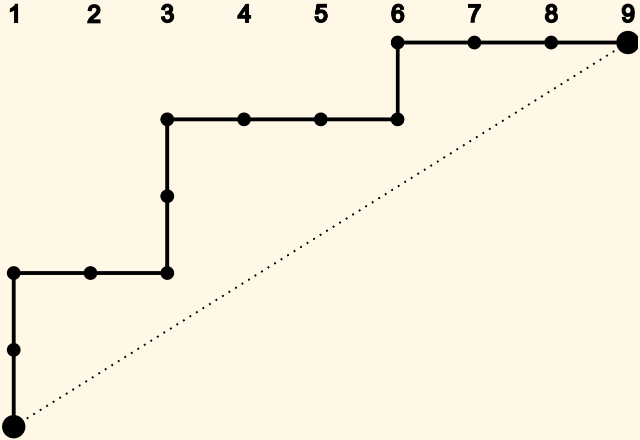
To define a "rational triangulation" . . .

- Start with a Dyck path. Here $(a, b) = (5, 8)$.



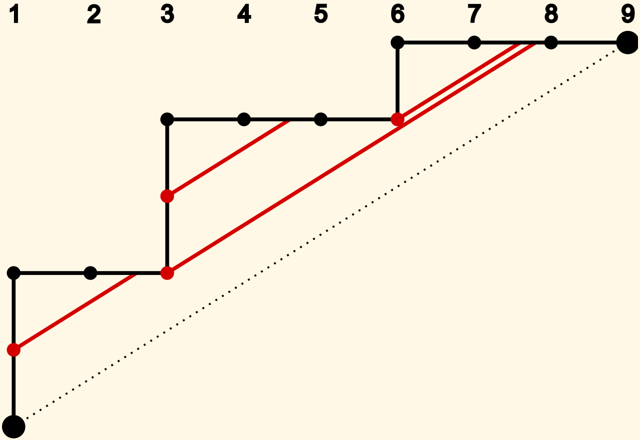
To define a "rational triangulation" ...

- Label the columns by $\{1, 2, \ldots, b+1\}$.



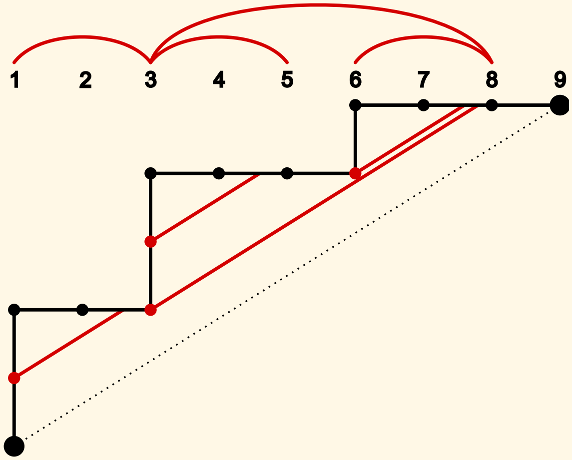
To define a "rational triangulation" . . .

- Shoot lasers from the bottom left with slope a/b.



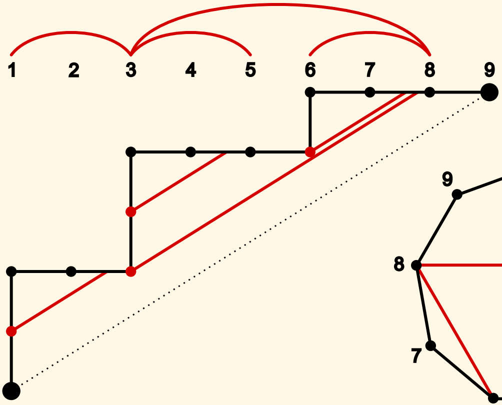
To define a “rational triangulation” ...

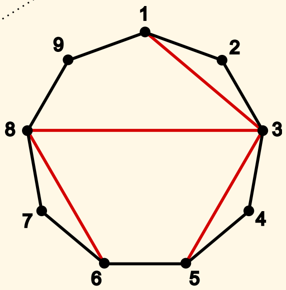
- Lift the lasers up.



To define a "rational triangulation" ...

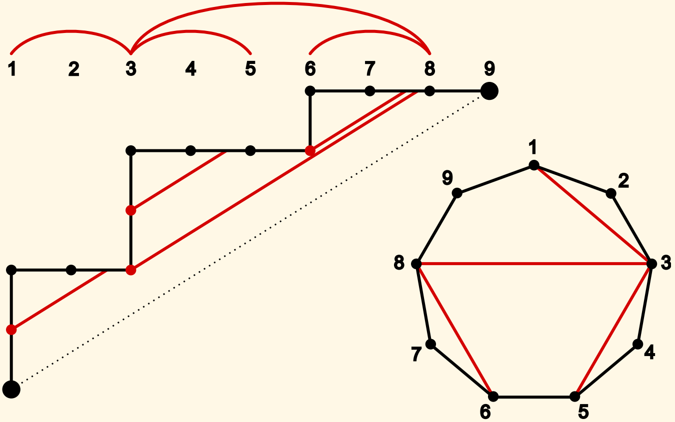
- There you go!





To define a “rational triangulation” . . .

- We have constructed $\mathrm{Cat}(a, b)$ many “rational triangulations” of a convex $(b + 1)$-gon, and each of them has $a - 1$ chords.



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Geometric Realization

Note that $\mathrm{Ass}(a, b)$ is a pure $(a-1)$-dimensional subcomplex of the $(b-1)$-dimensional polytope $\mathrm{Ass}(b-1)$.

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Theorems (with B. Rhoades and N. Williams)

- $\mathrm{Ass}(n, n+1)$ is the **classical associahedron** $\mathrm{Ass}(n)$.
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- $\mathrm{Ass}(x)$ is shellable and hence homotopy equivalent to a wedge of $\mathrm{Cat}'(x)$ many $(a-1)$-dimensional spheres.
- $\mathrm{Ass}(x)$ has h-vector $\mathrm{Nar}(x; k) = \frac{1}{a}\binom{a}{k}\binom{b-1}{k-1}$.
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Given $0 < x = a/(b-a)$ with $0 < a < b$ coprime, note that $\mathrm{Ass}(x) = \mathrm{Ass}(a, b)$ and $\mathrm{Ass}(1/x) = \mathrm{Ass}(b-a, b)$ are both subcomplexes of the polytope $\mathrm{Ass}(b-1) = \mathrm{Ass}(b-1, b)$.

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How do $\mathrm{Ass}(x)$ and $\mathrm{Ass}(1/x)$ fit together?

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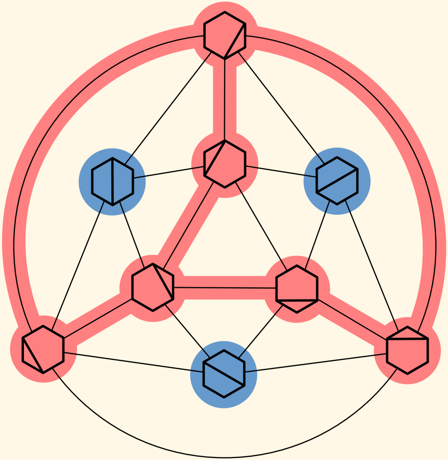
- Example: Here are subcomplexes $\mathsf{Ass}(2,5)$ and $\mathsf{Ass}(3,5)$ in $\mathsf{Ass}(4)$.



Rational Duality?

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Note that $\mathrm{Ass}(b-1)$ has this many vertices:

$$\binom{b+1}{2} - (b+1) = \frac{(b+1)b}{2} - \frac{2(b+1)}{2} = \frac{(b-2)(b+1)}{2}.$$

The subcomplexes $\mathrm{Ass}(a, b)$ and $\mathrm{Ass}(b-a, b)$ **bipartition** the vertices:

$$\frac{(a-1)(b+1)}{2} + \frac{(b-a-1)(b+1)}{2} = \frac{(b-2)(b+1)}{2}.$$

Rational Duality = Alexander Duality

Conjecture (with B. Rhoades and N. Williams)

We know that $\mathrm{Ass}(a, b)$ *and* $\mathrm{Ass}(b-a, b)$ *have the same number of homotopy spheres (of complementary dimensions) because*

$$\mathrm{Cat}'(a, b) = \mathrm{Cat}'(b-a, b).$$

We conjecture that the homotopy spheres are "intertwined" in a nice way. Formally, we conjecture that $\mathrm{Ass}(a, b)$ *and* $\mathrm{Ass}(b-a, b)$ *are* **Alexander dual** *as subcomplexes of* $\mathrm{Ass}(b-1)$.

Theorem (B. Rhoades)

The conjecture is true.

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$$\#\ \textbf{homotopy spheres}\ \mathrm{Ass}(a,b) = \#\ \textbf{maximal faces}\ \mathrm{Ass}'(a,b).$$

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What does the following mean?

$$\mathrm{Ass}(a,b) \mapsto \mathrm{Ass}'(a,b) \mapsto \mathrm{Ass}''(a,b) \mapsto \cdots \mapsto \ \textbf{a point}$$

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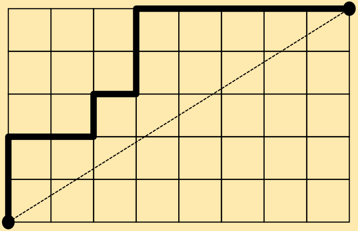
Next: Rational Parking Functions

The Rational Parking Space

Definition

- Label the up-steps by $\{1, 2, \ldots, a\}$, increasing up columns.



- Call this a **parking function**.
- Let $\mathrm{PF}(x) = \mathrm{PF}(a, b)$ denote the set of parking functions.
- **Classical form** $(z_1, z_2, \ldots, z_a)$ has label z_i in column i.
- Example: $(3, 1, 4, 4, 1)$

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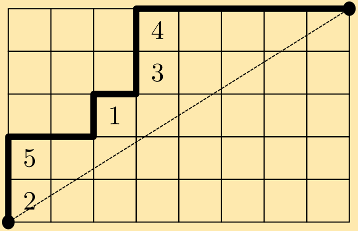

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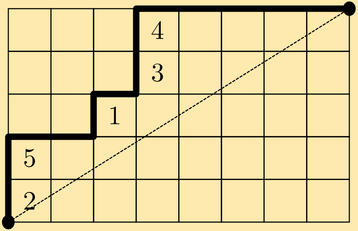

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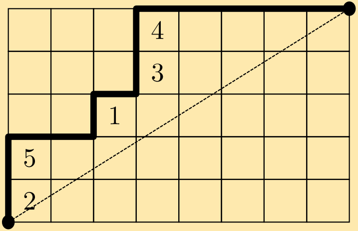

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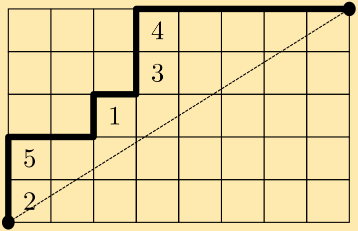

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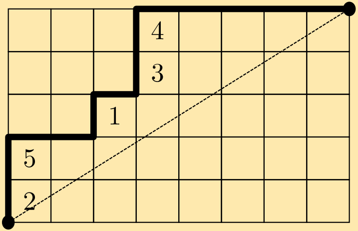

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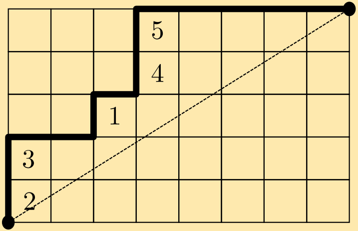

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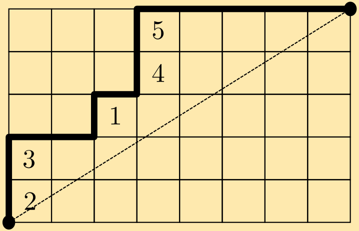

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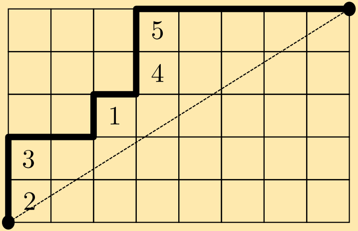

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- The dimension of $\mathrm{PF}(a,b)$ is b^{a-1}.
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What does switching $a \leftrightarrow b$ mean?

Problem

Given a, b coprime we have an $\mathfrak{S}_a$-module $\mathrm{PF}(a, b)$ of dimension b^{a-1} and an $\mathfrak{S}_b$-module $\mathrm{PF}(b, a)$ of dimension a^{b-1}.

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Summary of Catalan Numerology

- The Kirkman/Narayana/Schröder numbers are equivalent. They contain information about rank. $(1 < k < a-1)$

$$\left.\begin{aligned} \mathrm{Kirk}(x;k) &= \tfrac{1}{a}\tbinom{a}{k}\tbinom{b+k-1}{k-1} \\ \mathrm{Nar}(x;k) &= \tfrac{1}{a}\tbinom{a}{k}\tbinom{b-1}{k-1} \\ \mathrm{Schr\ddot{o}}(x;k) &= \tfrac{1}{b}\tbinom{a-1}{k}\tbinom{b+k}{a} \end{aligned}\right\} \quad \begin{aligned} &f\text{-vector} \\ &h\text{-vector} \\ &\text{“dual” } f\text{-vector} \end{aligned}$$

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Vielen Dank!

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