Math 310 **Exam 2**
Spring 2024 **Thurs Apr 25**

No electronic devices are allowed. No collaboration is allowed. There are 5 pages and each page is worth 6 points, for a total of 30 points.

1. Integrating a Scalar Over a Region in the Plane.

(a) Integrate $f(x, y) = xy$ over the rectangle with $0 \le x \le 2$ and $0 \le y \le 1$.

$$\begin{aligned}\int_0^1 \left(\int_0^2 xy\,dx\right) dy &= \int_0^1 \left[\frac{1}{2}x^2 y\right]_0^2 dy \\ &= \int_0^1 2y\,dy \\ &= [y^2]_0^1 \\ &= 1.\end{aligned}$$

(b) Integrate $f(x, y) = x^2 + y^2$ over the unit circle $x^2 + y^2 \le 1$. [Hint: Use polar coordinates.]

The unit circle is parametrized by $x = r\cos\theta$ and $y = r\sin\theta$ with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.Since $x^2 + y^2 = r^2$ and $dxdy = r\,drd\theta$ we have

$$\begin{aligned}\iint_{\text{circle}} (x^2 + y^2)\,dxdy &= \iint r^2\,r\,drd\theta \\ &= \int_0^{2\pi} \cdot \int_0^1 r^3\,dr \\ &= 2\pi \left[\frac{1}{4}r^4\right]_0^1 \\ &= \pi/2.\end{aligned}$$

2. Cylindrical and Spherical Coordinates.

(a) Use cylindrical coordinates to integrate the function $f(x, y, z) = z$ over the cylinder defined by $0 \le r \le 1$, $0 \le \theta \le 2\pi$ and $0 \le z \le 1$.

Since $dxdydz = r\,drd\theta dz$ we have

$$\begin{aligned}\iiint_{\text{cylinder}} z\,dxdydz &= \iiint zr\,drd\theta dz \\ &= \int_0^{2\pi} d\theta \cdot \int_0^1 r\,dr \cdot \int_0^1 z\,dz \\ &= 2\pi \cdot \left[\frac{1}{2}r^2\right]_0^1 \cdot \left[\frac{1}{2}z^2\right]_0^1 \\ &= \pi/2.\end{aligned}$$

(b) Use spherical coordinates to compute the **volume of a sphere of radius** 1. [Hint: You can take $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$.]

Since $dxdydz = \rho^2 \sin\phi \, d\rho d\theta d\phi$ we have

$$\begin{aligned}
\text{Volume} &= \iiint_{\text{sphere}} 1 \, dxdydz \\
&= \iiint \rho^2 \sin\phi \, d\rho d\theta d\phi \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 \, d\rho \cdot \int_0^{\pi} \sin\phi \, d\phi \\
&= 2\pi \cdot \left[\frac{1}{3}\rho^3\right]_0^1 \cdot [-\cos\phi]_0^{\pi} \\
&= 2\pi \cdot \frac{1}{3} \cdot [-(-1) + 1] \\
&= 4\pi/3.
\end{aligned}$$

3. Surface Area. Consider the following parametrized surface in 3D:

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle \qquad \text{with } 0 \le u \le 1 \text{ and } 0 \le v \le 2\pi.$$

(a) Compute the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$, and the normal vector $\mathbf{r}_u \times \mathbf{r}_v$.

We have

$$\begin{aligned}
\mathbf{r}_u &= \langle \cos v, \sin v, 0 \rangle, \\
\mathbf{r}_v &= \langle -u \sin v, u \cos v, 1 \rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{pmatrix} \\
&= \langle \sin v, -\cos v, u \cos^2 v + u \sin^2 v \rangle \\
&= \langle \sin v, -\cos v, u \rangle.
\end{aligned}$$

(b) Use your answer from part (a) to set up an integral to compute the **area of the surface** and simplify as much as possible. [This integral is too difficult to evaluate by hand; at least, for me it is.]

The surface area is

$$\begin{aligned}
\iint_{\text{surface}} 1 \, dA &= \iint 1 \, \|\mathbf{r}_u \times \mathbf{r}_v\| \, dudv \\
&= \iint \sqrt{\sin^2 v + \cos^2 v + u^2} \, dudv \\
&= \iint \sqrt{1 + u^2} \, dudv \\
&= \int_0^{2\pi} dv \cdot \int_0^1 \sqrt{1 + u^2} \, du
\end{aligned}$$

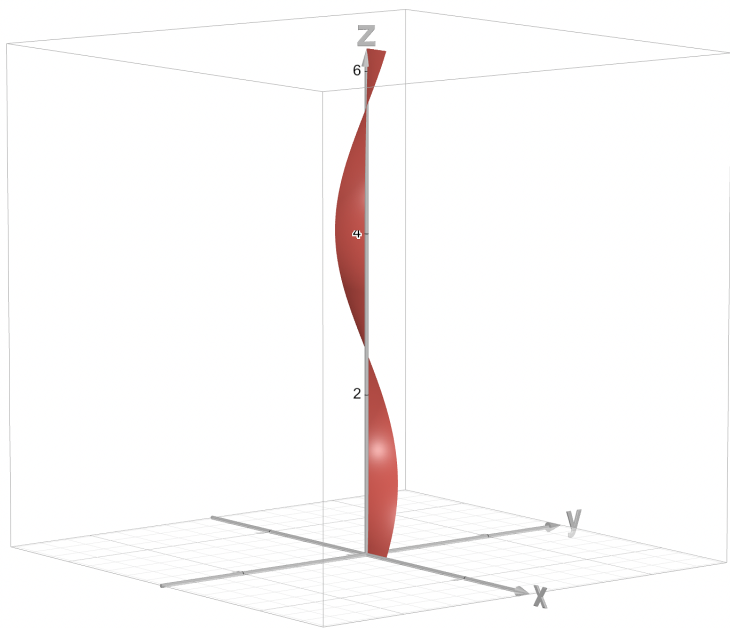
$$= 2\pi \cdot \int_0^1 \sqrt{1+u^2}\, du.$$

This last integral is not easy to compute. My computer says that

$$\int_0^1 \sqrt{1+u^2}\, du = \frac{1}{2}\left(\sqrt{2} + \ln(1+\sqrt{2})\right),$$

so the area of the surface is $\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \approx 7.2$.

Remark: This surface is called a *helicoid*. It looks like a twisted ribbon of width 1 and length 2π:



The untwisted ribbon would have surface area $2\pi \approx 6.8$. The twisted ribbon has been stretched so its area is slightly larger.

4. Conservative Vector Fields. Consider the vector field $\mathbf{F}(x, y, z) = \langle y, x + z, y\rangle$.

(a) Compute the curl $\nabla \times \mathbf{F}$ in order to verify that the field $\mathbf{F}$ is conservative. You need to show the steps of the computation, not just the final answer.

Let $\mathbf{F} = \langle P, Q, R\rangle = \langle y, x + z, y\rangle$. The curl is

$$\begin{aligned}
\nabla \times F &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \\
&= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle \\
&= \langle (y)_y - (x+z)_z, (y)_z - (y)_z, (x+y)_x - (y)_y\rangle \\
&= \langle 1-1, 0-0, 1-1\rangle \\
&= \langle 0, 0, 0\rangle,
\end{aligned}$$

which implies that $\mathbf{F}$ is conservative.

(b) Find a specific scalar function $f(x,y,z)$ such that $\nabla f(x,y,z) = \mathbf{F}(x,y,z)$. [Hint: Integrate $\mathbf{F}$ along any path in the plane that ends at the fixed point (x,y,z).]

The simplest choice is $\mathbf{r}(t) = \langle tx, ty, tz\rangle$ with $0 \le t \le 1$. Then we can take

$$\begin{aligned} f(x,y,z) &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt \\ &= \int_0^1 \mathbf{F}(tx,ty,tz) \bullet \langle x,y,z\rangle\, dt \\ &= \int_0^1 \langle ty, tx+tz, ty\rangle \bullet \langle x,y,z\rangle\, dt \\ &= \int_0^1 \left((ty)x + (tx+tz)y + (ty)z\right)\, dz \\ &= (yx+xy+zy+yz)\int_0^1 t\, dt \\ &= 2(xy+yz)\left[\frac{1}{2}t^2\right]_0^1 \\ &= xy+yz. \end{aligned}$$

Check:

$$\nabla(xy+yz) = \langle (xy+yz)_x, (xy+yz)_y, (xy+yz)_z\rangle = \langle y, x+z, y\rangle.$$

5. Green's Theorem. Consider the vector field $\mathbf{F}(x,y) = \langle -y+e^x, x+e^y\rangle$ in the plane.

(a) Compute the scalar function Curl($\mathbf{F}$).

Let $\mathbf{F} = \langle P, Q\rangle = \langle -y+e^x, x+e^y\rangle$. Then

$$\begin{aligned} \mathrm{Curl}(\mathbf{F}) &= Q_x - P_y \\ &= (x+e^y)_x - (-y+e^x)_y \\ &= 1-(-1) \\ &= 2. \end{aligned}$$

(b) Compute the integral of Curl($\mathbf{F}$) over the unit disk $x^2+y^2 \le 1$.

This can be solved using polar coordinates, but I will use the fact that I already know the area of the unit disk:

$$\iint_{\text{disk}} \mathrm{Curl}(\mathbf{F})\, dxdy = \iint_{\text{disk}} 2\, dxdy = 2\int_{\text{disk}} 1\, dxdy = 2\pi.$$

(c) Compute the integral of $\mathbf{F}$ around the curve $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \le t \le 2\pi$. [Hint: This integral is too difficult to compute directly, but there is a shortcut.]

If D is the unit disk then $\mathbf{r}(t)$ is the boundary ∂D. Combining Green's Theorem and part (b) gives

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt = \int_{\partial D} \mathbf{F} \bullet \mathbf{T} = \iint_D \mathrm{Curl}(\mathbf{F})\, dxdy = 2\pi.$$