Book Problems:

- Section 3.7, Exercises 4, 10, 20, 42
- Section 4.3, Exercises 4, 8, 12, 20, 38
- Section 4.4., Exercises 6, 10, 12
- Section 4.5, Exercises 2, 4, 8, 16, 20

3.7.4. Find the most general antiderivative of $f(x) = \sqrt[3]{x^2} + x\sqrt{x}$. First we express the function in terms of powers, then we use the backwards power rule:

$$\begin{aligned}\int f(x)\,dx &= \int (\sqrt[3]{x^2} + x\sqrt{x})\,dx \\ &= \int (x^{2/3} + x^{3/2})\,dx \\ &= \frac{x^{2/3+1}}{2/3+1} + \frac{x^{3/2+1}}{3/2+1} + c \\ &= \frac{x^{5/3}}{5/3} + \frac{x^{5/2}}{5/2} + c \\ &= \frac{3}{5}x^{5/3} + \frac{2}{5}x^{5/2} + c.\end{aligned}$$

Remark: The "$+c$" is what they mean by "most general".

3.7.10. Find the most general antiderivative of $f(t) = 3\cos t - 4\sin t$.

$$\begin{aligned}\int f(t)\,dt &= \int (3\cos t - 4\sin t)\,dt \\ &= 3(\sin t) - 4(-\cos t) + c \\ &= 3\sin t + 4\sin t + c.\end{aligned}$$

3.7.20. Find $f(x)$, where $f''(x) = 6x + \sin x$. We need to integrate twice. First we have

$$f'(x) = \int f''(x)\,dx = \int (6x + \sin x)\,dx = 6\frac{x^2}{2} - \cos x + c = 3x^2 - \cos x + c_1$$

for some constant c_1. Then we have

$$\begin{aligned}f(x) &= \int f'(x)\,dx \\ &= \int (3x^2 - \cos x + c_1)\,dx \\ &= 3 \cdot \frac{x^3}{3} - \sin x + c_1 x + c_2 \\ &= x^3 - \sin x + c_1 x + c_2\end{aligned}$$

for some constants c_1 and c_2.

3.7.42. Find $s(t)$ assuming that $s''(t) = t^2 - 4t + 6$, $s(0) = 0$ and $s(1) = 20$. First we integrate twice to obtain $s(t)$:

$$\begin{aligned} s'(t) &= \int s''(t)\,dt \\ &= \int (t^2 - 4t + 6)\,dt \\ &= \frac{1}{3}t^3 - 4 \cdot \frac{1}{2}t^2 + 6t + c_1 \\ &= \frac{1}{3} \cdot t^3 - 2 \cdot t^2 + 6t + c_1, \end{aligned}$$

and

$$\begin{aligned} s(t) &= \int s'(t)\,dt \\ &= \int \left(\frac{1}{3} \cdot t^3 - 2 \cdot t^2 + 6t + c_1\right)\,dt \\ &= \frac{1}{3} \cdot \frac{1}{4}t^4 - 2 \cdot \frac{1}{3}t^3 + 6 \cdot \frac{1}{2}t^2 + c_1 t + c_2, \end{aligned}$$

for some constants c_1 and c_2. To find the specific values of the constants we substitute $t = 0$ and $t = 1$ into our formula and use the given information:

$$\begin{aligned} 0 &= s(0) \\ 0 &= \frac{1}{3} \cdot \frac{1}{4}0^4 - 2 \cdot \frac{1}{3}0^3 + 6 \cdot \frac{1}{2}0^2 + c_1 0 + c_2 \\ 0 &= c_2, \end{aligned}$$

and

$$\begin{aligned} 20 &= s(1) \\ 20 &= \frac{1}{3} \cdot \frac{1}{4}1^4 - 2 \cdot \frac{1}{3}1^3 + 6 \cdot \frac{1}{2}1^2 + c_1 1 + c_2 \\ 20 &= \frac{1}{12} - \frac{2}{3} + 3 + c_1 + 0 \\ c_1 &= 20 - \frac{1}{12} + \frac{2}{3} - 3 \\ c_1 &= \frac{211}{12}. \end{aligned}$$

This looks wrong but my computer confirmed that it is right. In summary, we have

$$s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t.$$

4.3.4. Evaluate the integral:

$$\begin{aligned} \int_0^3 (1 + 6w^2 - 10w^4)\,dw &= \left[w + 6 \cdot \frac{1}{3}w^3 - 10 \cdot \frac{1}{5}w^5\right]_0^3 \\ &= \left[w + 2w^3 - 2w^5\right]_0^3 \\ &= \left[3 + 2 \cdot 3^3 - 2 \cdot 3^5\right] - \left[0 + 2 \cdot 0^3 - 2 \cdot 0^5\right] \\ &= -429. \end{aligned}$$

This also looks wrong, but my computer confirms it.

4.3.8. Evaluate the integral:

$$\begin{aligned}\int_1^2 \left(\frac{1}{x^2} - \frac{4}{x^3}\right) dx &= \int_1^2 \left(x^{-2} - 4x^{-3}\right) dx \\ &= \left[\frac{x^{-1}}{-1} - 4 \cdot \frac{x^{-2}}{-2}\right]_1^2 \\ &= \left[-\frac{1}{x} + \frac{2}{x^2}\right]_1^2 \\ &= \left[-\frac{1}{2} + \frac{2}{4}\right] - \left[-\frac{1}{1} + \frac{2}{1}\right] \\ &= -1.\end{aligned}$$

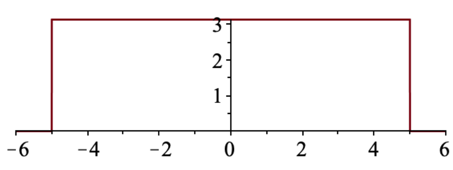
4.3.12. Evaluate the integral:

$$\begin{aligned}\int_0^1 (3 + x\sqrt{x})\, dx &= \int_0^1 (3 + x^{3/2})\, dx \\ &= \left[3x + \frac{x^{5/2}}{5/2}\right]_0^1 \\ &= \left[3x + \frac{2}{5}x^{5/2}\right]_0^1 \\ &= \left[3 + \frac{2}{5}\right] - 0 \\ &= \frac{17}{5}.\end{aligned}$$

4.3.20. Evaluate the integral:

$$\int_{-5}^{5} \pi\, dx = [\pi x]_{-5}^{5} = \pi(5) - \pi(-5) = 10\pi.$$

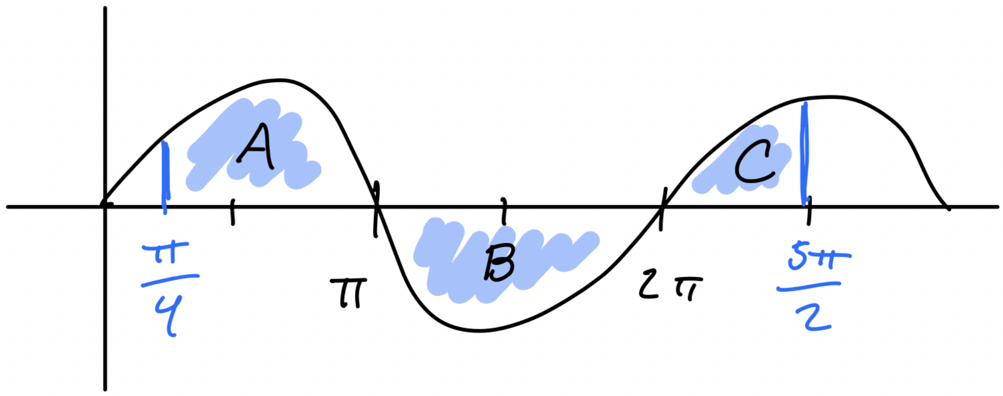
Remark: The graph of $f(x) = \pi$ is just a horizontal line at height π. The integral of this function from -5 to 5 is just the area of a rectangle of width 10 and height π:



4.3.38. Evaluate the integral:

$$\int_{\pi/4}^{5\pi/2} \sin x\, dx = [-\cos x]_{\pi/4}^{5\pi/2} = [-\cos(5\pi/2)] - [-\cos(\pi/4)] = 0 + 1/\sqrt{2} = 1/\sqrt{2}.$$

Illustrate with a sketch. Okay, I will:



If A, B, C are the areas of the three labeled regions, the computed integral is $A - B + C$.

4.4.6. Use the FTC to compute the derivative:

$$g(x) = \int_1^x (2 + t^4)^5 \, dt,$$
$$g'(x) = (2 + x^4)^5.$$

4.4.10. Use the FTC to compute the derivative of

$$h(x) = \int_1^{x^2} \sqrt{1 + r^3} \, dr.$$

I would rather compute the derivative of

$$g(x) = \int_1^x \sqrt{1 + r^3} \, dr,$$

which is $g'(x) = \sqrt{1 + x^3}$ because of the FTC. But we need to compute the derivative of

$$h(x) = \int_1^{x^2} \sqrt{1 + r^3} \, dr = g(x^2),$$

which we can do using the chain rule:

$$[g(x^2)]' = g'(x^2)(2x) = \sqrt{1 + (x^2)^3} \cdot (2x).$$

4.4.12. Compute the derivative of

$$y = \int_{\sin x}^1 \sqrt{1 + t^2} \, dt.$$

This is just the FTC but it's twisted twice. To untangle it, consider the nicer function

$$f(x) = \int_1^x \sqrt{1 + t^2} \, dt,$$

which has derivative $f'(x) = \sqrt{1 + x^2}$. But note that

$$y = -\int_1^{\sin x} \sqrt{1 + t^2} \, dt = -f(\sin x).$$

Then using the chain rule gives

$$\frac{dy}{dx} = [-f(\sin x)]' = -f'(\sin x) \cdot (\sin x)' = -\sqrt{1 + (\sin x)^2} \cdot (\cos x).$$

4.5.2. Evaluate the indefinite integral[1]

$$\int x^3 (2 + x^4)^5 \, dx.$$

by making the substitution $u = 2 + x^4$. First note that $du = (0 + 4x^3)dx$. Then we have

$$\begin{aligned}
\int x^3 (2 + x^4)^5 \, dx &= \int x^3 u^5 \left(\frac{du}{4x^3}\right) \\
&= \frac{1}{4} \int u^5 \, du \\
&= \frac{1}{4} \cdot \frac{1}{6} u^6 + c \\
&= \frac{1}{24} (2 + x^4)^6 + c.
\end{aligned}$$

4.5.4. Evaluate the indefinite integral

$$\int \frac{dt}{(1 - 6t)^4}$$

by making the substitution $u = 1 - 6t$. First note that $du = -6dt$. Then we have

$$\begin{aligned}
\int \frac{dt}{(1 - 6t)^4} &= \int \frac{-du/6}{u^4} \\
&= -\frac{1}{6} \int u^{-4} \, du \\
&= -\frac{1}{6} \cdot \frac{u^{-3}}{-3} + c \\
&= \frac{1}{18} u^{-3} + c \\
&= \frac{1}{18(1 - 6t)^3} + c.
\end{aligned}$$

4.5.8. Evaluate the indefinite integral

$$\int x^2 \cos(x^3) \, dx.$$

This time I have to choose the substitution myself. I will use $u = x^3$, so that $du = 3x^2 dx$ and $dx = du/(3x^2)$. Then we have

$$\begin{aligned}
\int x^2 \cos(x^3) \, dx &= \int x^2 \cos u \left(\frac{du}{3x^2}\right) \\
&= \frac{1}{3} \int \cos u \, du \\
&= \frac{1}{3} \sin u + c
\end{aligned}$$

[1] "Indefinite integral" just means "antiderivative".

$$= \frac{1}{3}\sin(x^3) + c.$$

4.5.16. Evaluate the indefinite integral

$$\int u\sqrt{1-u^2}\,du.$$

Oops, the letter u is already used so I will define $w = 1 - u^2$. Then $dw = (0 - 2u)du$ and $du = -dw/(2u)$, hence

$$\begin{aligned}
\int u\sqrt{1-u^2}\,du &= \int u\sqrt{w}\left(\frac{-dw}{2u}\right) \\
&= -\frac{1}{2}\int \sqrt{w}\,dw \\
&= -\frac{1}{2}\int w^{1/2}\,dw \\
&= -\frac{1}{2}\cdot\frac{w^{3/2}}{3/2} + c \\
&= -\frac{1}{2}\cdot\frac{2}{3}w^{3/2} + c \\
&= -\frac{w^{3/2}}{3} + c \\
&= -\frac{(1-u^2)^{3/2}}{3} + c
\end{aligned}$$

4.5.20. Evaluate the indefinite integral

$$\int (\cos\theta)^4 \sin\theta\,d\theta.$$

Let $u = \cos\theta$ so that $du = -\sin\theta d\theta$ and $d\theta = -du/\sin\theta$. Then we have

$$\begin{aligned}
\int (\cos\theta)^4 \sin\theta\,d\theta &= \int u^4 \sin\theta\left(-\frac{du}{\sin\theta}\right) \\
&= -\int u^4\,du \\
&= -\frac{u^5}{5} + c \\
&= -\frac{(\cos\theta)^5}{5} + c.
\end{aligned}$$