Math 161 **Summer 2024**
Homework 3 **Drew Armstrong**

Book Problems:

Problems from Section 3.3 are worth two points each. All other problems are worth one point.

- Section 2.7, Exercises 2, 8, 10
- Section 2.8, Exercises 4, 12, 22, 26
- Section 3.3., Exercises 26, 30, 32
- Section 3.5, Exercises 2, 8, 12, 16

2.7.2. (a): If A is the area and r is the radius of a circle then $A = \pi r^2$. If A and r are both functions of time t then applying $\frac{d}{dt}$ to both sides gives

$$\begin{aligned} A &= \pi r^2 \\ \frac{d}{dt}A &= \pi \frac{d}{dt}(r^2) \\ \frac{dA}{dt} &= \pi(2r)\frac{dr}{dt} \\ \frac{dA}{dt} &= 2\pi r \cdot \frac{dr}{dt}. \end{aligned}$$

(b): In this word problem we are given that $dr/dt = 1$ and $r = 30$. Then using the formula from part (a) gives

$$\frac{dA}{dt} = 2\pi(30)(1) = 60\pi.$$

2.7.8. Suppose $4x^2 + 9y^2 = 36$, where x and y are functions of t. Applying d/dt to both sides gives[1]

$$\begin{aligned} (4x^2 + 9y^2)' &= (36)' \\ 4(x^2)' + 9(y^2)' &= 0 \\ 4(2xx') + 9(2yy') &= 0 \\ 8xx' + 18yy' &= 0. \end{aligned}$$

(a): If $y' = 1/3$ and $x = 2$ and $y = \frac{2}{3}\sqrt{5}$ then we have

$$\begin{aligned} 8xx' + 18yy' &= 0 \\ 8(2)x' + 18(\frac{2}{3}\sqrt{5})(1/3) &= 0 \\ 16x' + 4\sqrt{5} &= 0 \\ x' &= -4\sqrt{5}/16 \\ x' &= -\sqrt{5}/4. \end{aligned}$$

(b): If $x' = 3$ and $x = -2$ and $y = \frac{2}{3}\sqrt{5}$ then

$$8xx' + 18yy' = 0$$

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[1]To save space we will use the "prime" symbol instead of d/dt.

$$\begin{aligned}8(-2)(3)+18(\tfrac{2}{3}\sqrt{5})y' &= 0\\ -48+12\sqrt{5}\,y' &= 0\\ y' &= 48/(12\sqrt{5})\\ y' &= 4/\sqrt{5}.\end{aligned}$$

Remark: Why are there fractions and radicals in this textbook problem? I don't know.

2.7.10. A particle is moving along a hyperbola $xy = 8$. As it reaches the point $(4, 2)$ the y-coordinate is decreasing at a rate of 3 cm/s. How fast is the x-coordinate changing at this instant?

We are given $dy/dt = -3$ and we want to find dx/dt when $x = 4$ and $y = 2$. First we apply d/dt to both sides of the equation $xy = 8$, assuming that x and y are both functions of time t:

$$\begin{aligned}(xy)' &= (8)'\\ x'y + xy' &= 0.\end{aligned}$$

When $y' = -3$ and $x = 4$ and $y = 2$ we have

$$\begin{aligned}x'y + xy' &= 0\\ x'(2) + (4)(-3) &= 0\\ 2x' &= 12\\ x' &= 6.\end{aligned}$$

In other words, $dx/dt = 6$ cm/s.

2.8.4. Find the linear approximation of $f(x) = x^{3/4}$ at $a = 16$.

The general formula says that $f(x) \approx f(a) + f'(a)(x - a)$ when $x \approx a$. First we compute the ingredients of the formula:

$$\begin{aligned}f(16) &= 16^{3/4} = (16^{1/4})^3 = 2^3 8,\\ f'(x) &= (3/4)x^{-1/4},\\ f'(16) &= (3/4)(16)^{-1/4} = (3/4)(16^{1/4})^{-1} = (3/4)(2^{-1}) = 3/8.\end{aligned}$$

Thus we conclude that

$$x^{3/4} \approx 8 + \frac{3}{8}(x - 16) \quad \text{when } x \approx 16.$$

2.8.12. Use linear approximation to estimate $\sqrt[3]{1001}$. The bad number 1001 is close to the good number $a = 1000$ and we wish to estimate $f(1001)$ when $f(x) = \sqrt[3]{x}$. The general formula says that

$$f(x) \approx f(1000) + f'(1000)(x - 1000) \quad \text{when } x \approx 1000.$$

Luckily we know that $f(1000) = \sqrt[3]{1000} = 10$ and $f'(x) = (1/3)x^{-2/3}$, hence

$$f'(1000) = \frac{1}{3}\cdot(1000)^{-2/3} = \frac{1}{3}\cdot(1000^{1/3})^{-2} = \frac{1}{3}\cdot(10)^{-2} = \frac{1}{300}.$$

Plugging this into the formula gives

$$\sqrt[3]{x} \approx 10 + \frac{1}{300}(x - 1000) \quad \text{when } x \approx 1000.$$

Finally, since 1001 is close to 1000 we have

$$\sqrt[3]{1001} \approx 10 + \frac{1}{300}(1001 - 1000) = 10 + \frac{1}{300} = 10.0033\cdots.$$

2.8.22. The radius of a dircular disk is given as 24 cm with a maximum error in measurement of 0.2 cm. Let A be the area and let r be the radius, so that $A = \pi r^2$ and

$$\begin{aligned} \frac{dA}{dr} &= 2\pi r \\ dA &= 2\pi r\, dr. \end{aligned}$$

(a): We are given that $r = 24$ and $dr = 0.2$, hence $dA = 2\pi(24)(0.2) \approx 30.2\ \text{cm}^2$.

(b): The relative (percentage) error in A is

$$\frac{dA}{A} = \frac{30.3}{\pi(24)^2} = 0.0167 \quad (= 1.7\%).$$

Remark: The relative (percent) error in r is $dr/r = 0.2/24 = 0.83\%$. The relative error always goes up when you perform a computation.

2.8.26. One side of a right triangle is known to be 20 cm long and the opposite angle is measured as 30°, with a possible error of ±1°.[2]

(a): If h is the length of the hypotenuse and θ is the angle whose opposite side has length 20 then by definition we have $\sin\theta = 20/h$, or $h = 20/\sin\theta$. We are given that $\theta = 30°$ and $d\theta = 1°$, but derivatives only work when we express angles in radians, so we must take

$$\theta = 30 \cdot \frac{2\pi}{360} = \frac{\pi}{6} \quad \text{and} \quad d\theta = 1 \cdot \frac{2\pi}{360} = \frac{\pi}{180}.$$

Then we compute[3]

$$\begin{aligned} h &= 20/\sin\theta \\ \frac{dh}{d\theta} &= \frac{-20\cos\theta}{\sin^2\theta} \\ dh &= \frac{-20\cos\theta}{\sin^2\theta} \cdot d\theta \\ dh &= \frac{-20\cos(30°)}{\sin^2(30°)} \cdot \frac{\pi}{180} \\ dh &= \frac{-20(\sqrt{3}/2)}{(1/2)^2} \cdot \frac{\pi}{180} \\ dh &= -40\sqrt{3} \cdot \frac{\pi}{180} \\ dh &\approx -1.21. \end{aligned}$$

(b): The percentage error in h is

$$\frac{dh}{h} = \frac{-1.21}{20/\sin(30°)} = \frac{-1.21}{40} \approx 3\%.$$

[2]The book doesn't tell us the error in the measurement 26 cm because it didn't yet teach us how to deal with two different inputs. See pages 651–653 for the method, which is in Chapter 11.

[3]So actually we don't need to express θ in radians, just $d\theta$.

3.3.26. Sketch the graph of $h(x) = 5x^3 - 3x^5$. First we compute the first and second derivatives:

$$\begin{aligned} h(x) &= 5x^3 - 3x^5, \\ h'(x) &= 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1 - x)(1 + x), \\ h''(x) &= 30x - 60x^3 = 30x(1 - 2x^2). \end{aligned}$$

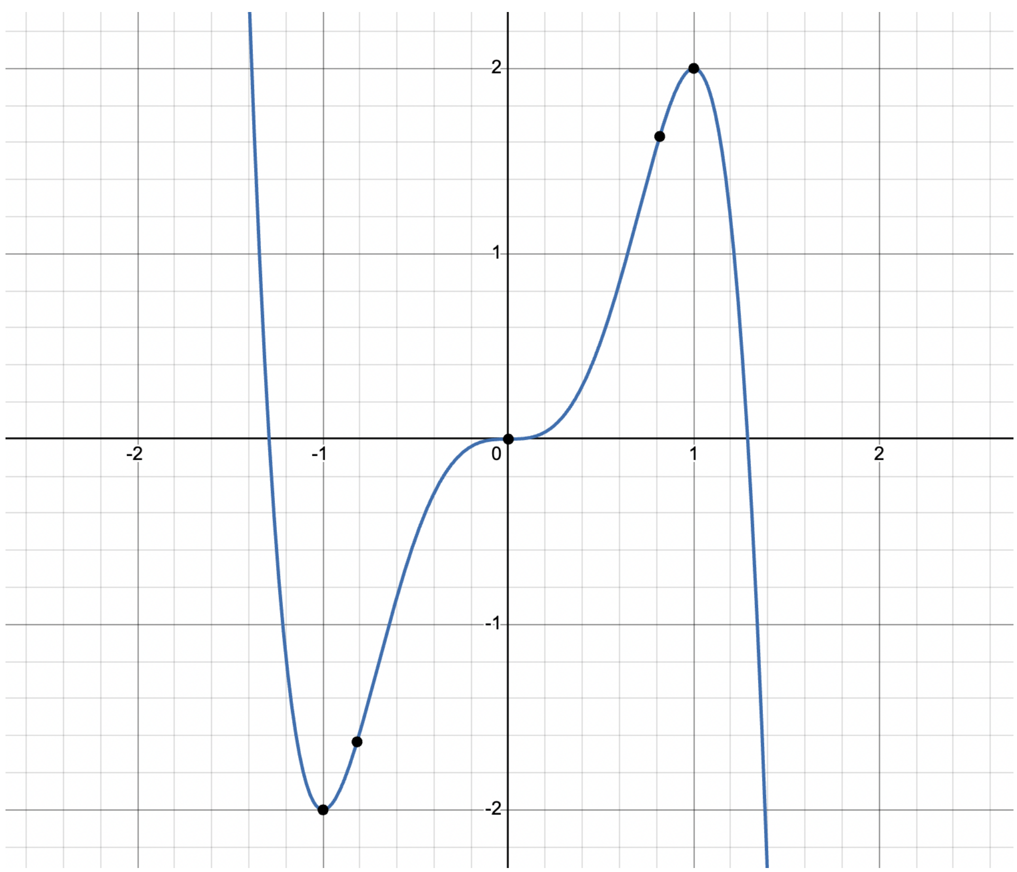
Examining the first derivative:

- $h'(x) = 0$ when $x = \pm 1$ or $x = 0$,
- $h'(x) > 0$ when $-1 < x < 0$ or $0 < x < 1$,
- $h'(x) < 0$ when $x < -1$ or $x > +1$.

This tells us that $h(x)$ has a local min at $x = -1$, an inflection at $x = 0$ and local max at $x = +1$. Examining the second derivative:

- $h''(x) = 0$ when $x = 0$ or $x = \pm\sqrt{1/2}$,
- $h''(x) > 0$ when $x < -\sqrt{1/2}$ or $0 < x < +\sqrt{1/2}$,
- $h''(x) < 0$ when $-\sqrt{1/2} < x < 0$ or $+\sqrt{1/2} < x$.

This tells us that $h(x)$ has inflections when $x = 0$ and $x = \pm\sqrt{1/2}$. Here is a picture with the special points marked:



3.3.30. Sketch the graph of $G(x) = x - 4\sqrt{x}$. First we compute the first and second derivatives:

$$G(x) = x - 4\sqrt{x},$$

$$G'(x) = 1 - 4 \cdot \frac{1}{2\sqrt{x}} = \frac{\sqrt{x} - 2}{\sqrt{x}}$$
$$G''(x) = -2(-1/2)x^{-3/2} = 1/x^{3/2}.$$

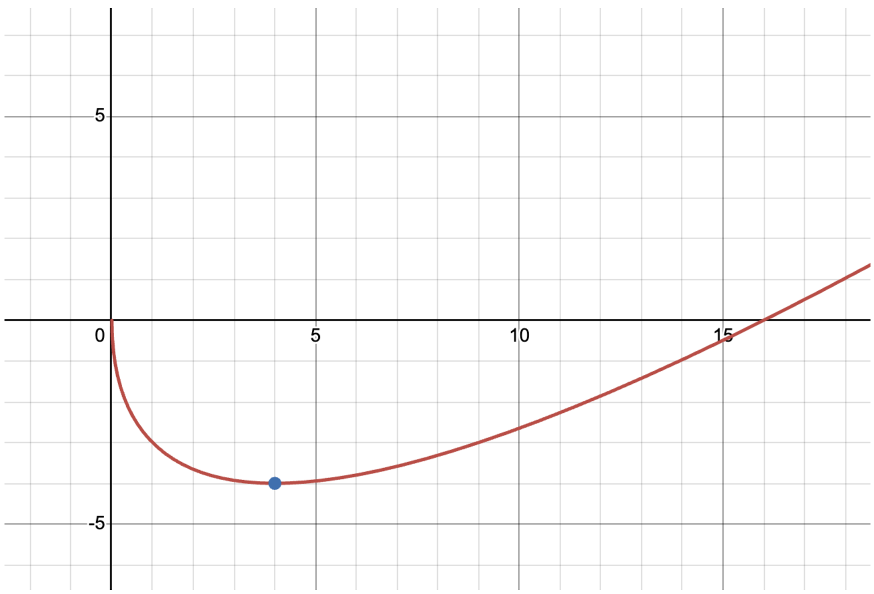
This function is only defined for $x \geq 0$. Note that the denominators of $G'(x)$ and $G''(x)$ are always positive so we only need to look at the numerators. Examining the first derivative:[4]

- $G'(x) = 0$ when $x = 4$,
- $G'(x) > 0$ when $4 < x$,
- $G'(x) < 0$ when $0 < x < 4$.

This tells us that $G(x)$ has a local min at $x = 4$. Examining the second derivative:

- $G''(x) = 0$ never,
- $G''(x) > 0$ always,
- $G''(x) < 0$ never.

Thus $G(x)$ has no inflections and is always concave up. Here is a picture:



3.3.32. Sketch the graph of $S(x) = x - \sin x$ for $0 \leq x \leq 4\pi$. First we compute the first and second derivatives:

$$S(x) = x - \sin x,$$
$$S'(x) = 1 - \cos x,$$
$$S''(x) = \sin x.$$

Examining the first derivative:

- $S'(x) = 0$ when $\cos x = 1$, i.e., when $x = 0$ or $x = 2\pi$ or $x = 4\pi$, etc.
- $S'(x) > 0$ for all other x,
- $S'(x) < 0$ never.

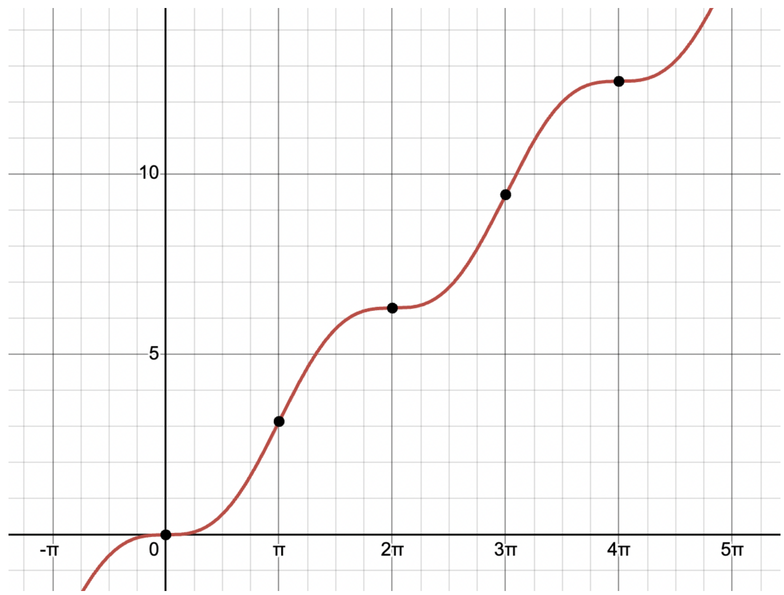
There are no maxima or minima. The graph of $S(x)$ is always increasing or flat. Examining the second derivative:

- $S''(x) = 0$ when $\sin x = 0$, i.e., when $x = \pi$ or $x = 2\pi$ or $x = 3\pi$, etc.

[4]We also note that $G'(x) \to -\infty$ as $x \to 0$ from the right, so the graph of $G(x)$ becomes vertical at $x = 0$.

- $S''(x) > 0$ when $\sin x > 0$, i.e., when $0 < x < \pi$ or $2\pi < x < 3\pi$, etc.
- $S''(x) <$ when $\sin x < 0$, i.e., when $\pi < x < 2\pi$ or $3\pi < x < 4\pi$, etc.

Thus we have inflections when $x = k\pi$ for any whole number k. In between, the graph of $S(x)$ alternates between concave up and down. Here is a picture:



3.5.2. Find two numbers whose difference is 100 and whose product is a minimum.

Call the numbers x and y, so that $x - y = 100$. We want to minimize the product $P(x, y) = xy$. Substituting $y = x - 100$ into P gives $P(x) = x(x - 100)$ as a function of x alone. To find maxima or minima of $P(x)$ we solve the equation $P'(x) = 0$. First we compute

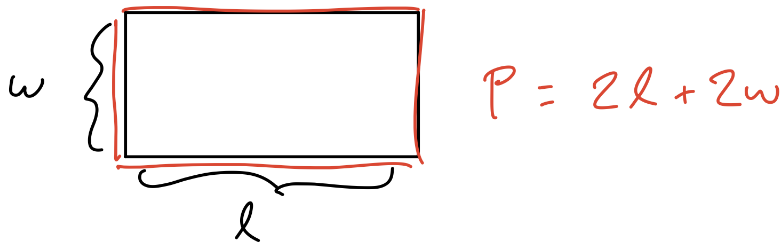
$$\begin{aligned} P(x) &= x(x-100) = x^2 - 100x, \\ P'(x) &= 2x - 100. \end{aligned}$$

Then we solve

$$\begin{aligned} P'(x) &= 0 \\ 2x - 100 &= 0 \\ x &= 50. \end{aligned}$$

(This gives a minimum of $P(x)$ because the second derivative $P''(x) = 100$ is always positive.) Hence $P = xy$ is minimized when $x = 50$ and $y = x - 100 = 50 - 100 = -50$.

3.5.8. Find the dimensions of a rectangle with area 1000 m^2 whose perimeter is as small as possible. If ℓ and w are the dimensions of the rectangle then the perimeter is $P = 2\ell + 2w$:



We want to minimize $P(\ell, w) = 2\ell + 2w$ subject to the constraint $\ell w = 1000$. First we use this constraint to eliminate w from P:

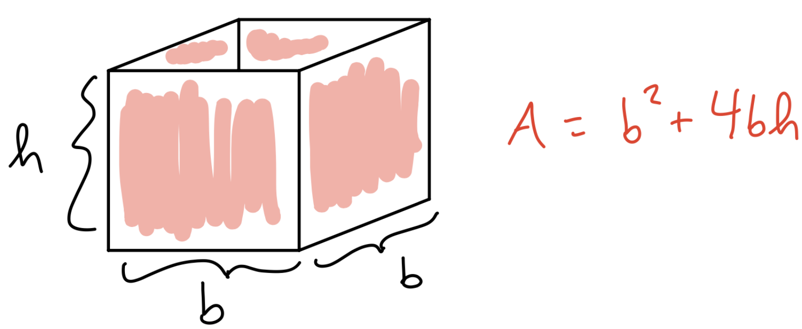
$$P(\ell) = 2\ell + 2w = 2\ell + 2(1000/\ell) = 2\ell + 2000/\ell.$$

Then to minimize P we set the first derivative equal to zero:

$$\begin{aligned} P'(\ell) &= 0 \\ 2 + 2000(-1/\ell^2) &= 0 \\ -2000/\ell^2 &= -2 \\ 1/\ell^2 &= 2/2000 \\ \ell^2 &= 1000 \\ \ell &= \sqrt{1000}. \end{aligned}$$

We conclude that P is minimized[5] when $\ell = \sqrt{1000}$, and hence $w = 1000/\sqrt{1000} = \sqrt{1000}$. In other words, for a given area the perimeter is maximized when the rectangle is a square.

3.5.12. A box with a square base and open top must have a volume of 32000 cm^3. Find the dimensions of the box that minimize the amount of material used (say, cardboard). Let b be the base and let h be the height of the box. The amount of cardboard is the surface area $A = b^2 + 4bh$:



In order to minimize A we first eliminate h using the volume constraint:

$$\begin{aligned} \text{volume} &= 32000 \\ b^2 h &= 32000 \\ h &= 32000/b^2. \end{aligned}$$

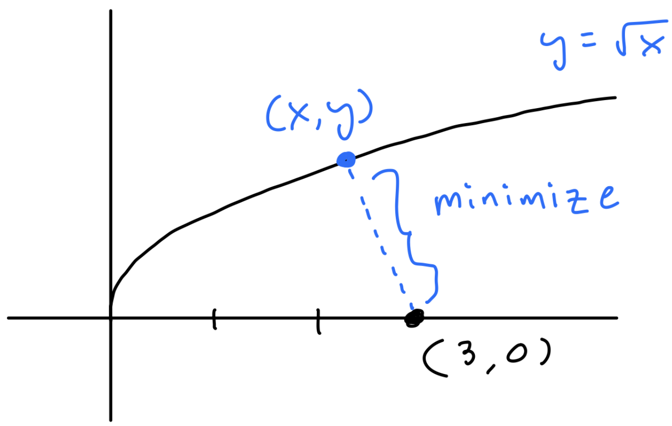
[5]To verify that this really is a minimum, consider the second derivative: $P''(\ell) = 4000/\ell^3$, which is always positive for positive ℓ. In particular, $P''(\sqrt{1000}) > 0$.

Hence we have $A = b^2 + 4bh = b^2 + 4b(32000/b^2) = b^2 + 128000/b$. Then to minimize A we set the first derivative equal to zero:

$$\begin{aligned} A'(b) &= 0 \\ 2b + 128000(-1/b^2) &= 0 \\ 2b^3 - 128000 &= 0 \\ 2b^3 &= 128000 \\ b^3 &= 64000 \\ b &= 40. \end{aligned}$$

We conclude that the amount of material is minimized when $b = 40$ and $h = 32000/40^2 = 20$.

3.5.16. Find the point (x, y) on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$:



The distance between **any two points** (x, y) and (a, b) is $\sqrt{(x-a)^2 + (y-b)^2}$. In particular, the distance between (x, y) and $(3, 0)$ is $D = \sqrt{(x-3)^2 + (y-0)^2} = \sqrt{x^2 - 6x + 9 + y^2}$. In order to minimize the distance we first use the constraint $y = \sqrt{x}$ to eliminate y from D:

$$D = \sqrt{x^2 - 6x + 9 + y^2} = \sqrt{x^2 - 6x + 9 + x} = \sqrt{x^2 - 5x + 9}.$$

Now we set the first derivative equal to zero:

$$\begin{aligned} D'(x) &= 0 \\ \frac{1}{2\sqrt{x^2 - 5x + 9}}(2x - 5 + 0) &= 0 \\ 2x - 5 &= 0 \\ x &= 5/2. \end{aligned}$$

(Here we used the fact that $a/b = 0$ implies $a = 0$ for any fraction.) We conclude that the distance D is minimized when $x = 5/2$ and $y = \sqrt{x} = \sqrt{5/2}$.