Math 161 **Summer 2024**
Homework 2 **Drew Armstrong**

Book Problems:

- Section 2.3, Exercises 2, 8, 14, 18, 20, 24
- Section 2.4, Exercises 4, 10, 14, 18, 24
- Section 2.5, Exercises 2, 12, 22, 25, 32
- Section 2.6, Exercises 3, 12, 14, 20

2.3.2. Differentiate $f(x) = \pi^2$. Since π^2 is just a constant we have $f'(x) = 0$.

2.3.8. Differentiate $y = \sin t + \pi \cos t$.

$$\frac{dy}{dt} = \frac{d}{dy}(\sin t + \pi \cos t) = \frac{d}{dt}(\sin t) + \pi \frac{d}{dt}(\cos t) = \cos t + \pi(-\sin t).$$

2.3.14. Differentiate $y = x^{5/3} - x^{2/3}$.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{5/3} - x^{2/3}\right) = \frac{d}{dx}x^{5/3} - \frac{d}{dx}x^{2/3} = \frac{5}{3}x^{2/3} + \frac{2}{3}x^{-1/3}.$$

2.3.18. Differentiate $S(R) = 4\pi R^2$.

$$S'(R) = 4\pi(2R) = 8\pi R.$$

2.3.20. Differentiate $g(u) = \sqrt{2}\,u + \sqrt{3u}$.

$$\begin{aligned} g'(u) &= \left(\sqrt{2}\,u\right)' + \left((3u)^{1/2}\right)' \\ &= \sqrt{2} + \frac{1}{2}(3u)^{-1/2}(3u)' \\ &= \sqrt{2} + \frac{1}{2}(3u)^{-1/2}(3). \end{aligned}$$

2.3.24. Differentiate $y = (\sin\theta)/2 + c/\theta$. I will assume that c is constant and θ is the variable, but the textbook isn't specific about this.

$$\begin{aligned} \frac{dy}{d\theta} &= \frac{d}{d\theta}\left(\frac{\sin\theta}{2} + \frac{c}{\theta}\right) \\ &= \frac{1}{2}\frac{d}{d\theta}(\sin\theta) + c\,\frac{d}{d\theta}\left(\frac{1}{\theta}\right) \\ &= \frac{1}{2}\cos\theta + c\left(-\frac{1}{\theta^2}\right). \end{aligned}$$

2.4.4. Differentiate $f(x) = \sqrt{x}\,\sin x$. We use the product rule:

$$\begin{aligned} f'(x) &= (\sqrt{x})'\,\sin x + \sqrt{x}\,(\sin x)' \\ &= \frac{1}{2\sqrt{x}}\,\sin x + \sqrt{x}\cos x. \end{aligned}$$

2.4.10. Differentiate $y = \sin\theta\cos\theta$. We use the product rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta}(\sin\theta)\ \cos\theta + \sin\theta\,\frac{d}{d\theta}(\cos\theta)\\ &= \cos\theta\ \cos\theta + \sin\theta\,(-\sin\theta)\\ &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

Alternatively, we can use the trig identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{d}{d\theta}\sin\theta\cos\theta\\ &= \frac{d}{d\theta}\frac{\sin(2\theta)}{2}\\ &= \frac{1}{2}\frac{d}{d\theta}\sin(2\theta)\\ &= \frac{1}{2}\ \cos(2\theta)\frac{d}{d\theta}(2\theta)\\ &= \frac{1}{2}\ \cos(2\theta)(2)\\ &= \cos(2\theta).\end{aligned}$$

This is the same answer because of the trig identity $\cos(2\theta) = \cos^2\theta - \sin^2\theta$.

2.4.14. Differentiate $y = \dfrac{x+1}{x^2+x-2}$. We use the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2+x-2)(x+1)' - (x+1)(x^2+x-2)'}{(x^2+x-2)^2}\\ &= \frac{(x^2+x-2)(1) - (x+1)(2x+1)}{(x^2+x-2)^2}.\end{aligned}$$

This can't be simplified very much.

2.4.18. Differentiate $y = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$. We use the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(\sqrt{x}+1)(\sqrt{x}-1)' - (\sqrt{x}-1)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2}\\ &= \frac{(\sqrt{x}+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x}-1)\left(\frac{1}{2\sqrt{x}}\right)}{(\sqrt{x}+1)^2}.\end{aligned}$$

There is no reason to simplify this, but it can be simplified to get

$$\frac{dy}{dx} = \frac{1}{\sqrt{x}\,(\sqrt{x}+1)^2}.$$

2.4.24. Differentiate $y = \dfrac{t}{(t-1)^2}$. We use the quotient rule:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t-1)^2(t)' - t[(t-1)^2]'}{[(t-1)^2]^2}\\ &= \frac{(t-1)^2(1) - t[2(t-1)(t-1)']}{(t-1)^4}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(t-1)^2(1) - t[2(t-1)(1)]}{(t-1)^4} \\
&= \frac{(t-1)^2 - 2t(t-1)}{(t-1)^4} \\
&= \frac{(t-1)\,[(t-1) - 2t]}{(t-1)^4} \\
&= \frac{(t-1) - 2t}{(t-1)^3} \\
&= \frac{-t-1}{(t-1)^3}.
\end{aligned}
$$

This can't be simplified any more.

2.5.2. Differentiate $y = (2x^3 + 5)^4$. We use the chain rule:

$$\frac{dy}{dx} = 4(2x^3+5)^3(2x^3+5)' = 4(2x^3+5)^3(6x^2).$$

2.5.12. Differentiate $f(t) = \sqrt[3]{1 + \tan t}$. It is helpful to compute the derivative of $\tan t$ first:

$$(\tan t)' = \left(\frac{\sin t}{\cos t}\right)' = \frac{\cos t(\sin t)' - \sin t(\cos t)'}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}.$$

(This is also a formula you can memorize.) Then we have

$$
\begin{aligned}
f'(t) &= \left[(1+\tan t)^{1/3}\right]' \\
&= \frac{1}{3}(1+\tan t)^{-2/3}(1+\tan t)' \\
&= \frac{1}{3}(1+\tan t)^{-2/3}\,\frac{1}{\cos^2 t}.
\end{aligned}
$$

My computer gives the answer

$$f'(t) = \frac{1+\tan^2 t}{3(1+\tan t)^{2/3}}.$$

This is the same because of the trig identity

$$1 + \tan^2 t = 1 + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}.$$

I don't have any of this memorized; I just work it out when I need to. The only trig identity you really need is $\sin^2 t + \cos^2 t = 1$.

2.5.22. Differentiate $f(s) = \sqrt{\dfrac{s^2+1}{s^2+4}}$. We use the chain rule and then the quotient rule:

$$
\begin{aligned}
f'(s) &= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-1/2}\left(\frac{s^2+1}{s^2+4}\right)' \\
&= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-1/2}\frac{(s^2+4)(s^2+1)' - (s^2+1)(s^2-4)'}{(s^2+4)^2} \\
&= \frac{1}{2}\left(\frac{s^2+1}{s^2+4}\right)^{-1/2}\frac{(s^2+4)(2s) - (s^2+1)(2s)}{(s^2+4)^2}.
\end{aligned}
$$

There is no reason to simplify this.

2.5.25. Differentiate $y = \dfrac{r}{\sqrt{r^2+1}}$. It is helpful to first compute

$$\left(\sqrt{r^2+1}\right)' = \frac{1}{2}(r^2+1)^{-1/2}(2r) = \frac{r}{\sqrt{r^2+1}}.$$

Then we use the quotient rule:

$$\begin{aligned}\frac{dy}{dr} &= \frac{\sqrt{r^2+1}\,(r)' - r(\sqrt{r^2+1})'}{(\sqrt{r^2+1})^2} \\ &= \frac{\sqrt{r^2+1}\,(1) - r\left(\frac{r}{\sqrt{r^2+1}}\right)}{(\sqrt{r^2+1})^2}.\end{aligned}$$

Does this simplify? Actually it does. Multiply top and bottom by $\sqrt{r^2+1}$ to get

$$\frac{dy}{dr} = \frac{\sqrt{r^2+1} - r\left(\frac{r}{\sqrt{r^2+1}}\right)}{(\sqrt{r^2+1})^2} \cdot \frac{\sqrt{r^2+1}}{\sqrt{r^2+1}} = \frac{(r^2+1)-r^2}{(r^2+1)(r^2+1)^{1/2}} = \frac{1}{(r^2+1)^{3/2}}.$$

It is useful to know this because it shows that $dy/dr > 0$ for all r. That is, the function y is always increasing.

2.5.32. Differentiate $y = x\,\sin\frac{1}{x}$. We use the product rule and the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x)\,\sin\frac{1}{x} + x\,\frac{d}{dx}\left(\sin\frac{1}{x}\right) \\ &= (1)\,\sin\frac{1}{x} + x\left(\cos\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \sin\frac{1}{x} + x\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \\ &= \sin\frac{1}{x} - \frac{1}{x}\left(\cos\frac{1}{x}\right).\end{aligned}$$

2.6.3. Find dy/dx by implicit differentiation.

$$\begin{aligned}x^3+y^3 &= 1 \\ \frac{d}{dx}(x^3+y^3) &= \frac{d}{dx}1 \\ 3x^2+3y^2\,\frac{dy}{dx} &= 0 \\ 3y^2\,\frac{dy}{dx} &= -3x^2 \\ \frac{dy}{dx} &= \frac{-3x^2}{3y^2} \\ \frac{dy}{dx} &= -\frac{x^2}{y^2}.\end{aligned}$$

2.6.12. Find dy/dx by implicit differentiation.

$$\sqrt{x+y} = 1 + x^2y^2$$

$$
\begin{aligned}
\frac{d}{dx}\sqrt{x+y} &= \frac{d}{dx}(1+x^2y^2)\\
\frac{1}{2}(x+y)^{-1/2}\left(1+\frac{dy}{dx}\right) &= 0+(2x)y^2+x^2(2y)\frac{dy}{dx}\\
\frac{1}{2}(x+y)^{-1/2}+\frac{1}{2}(x+y)^{-1/2}\frac{dy}{dx} &= 2xy^2+2x^2y\frac{dy}{dx}\\
\frac{1}{2}(x+y)^{-1/2}\frac{dy}{dx}-2x^2y\frac{dy}{dx} &= 2xy^2-\frac{1}{2}(x+y)^{-1/2}\\
\left(\frac{1}{2}(x+y)^{-1/2}-2x^2y\right)\frac{dy}{dx} &= 2xy^2-\frac{1}{2}(x+y)^{-1/2}\\
\frac{dy}{dx} &= \frac{2xy^2-\frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2}-2x^2y}.
\end{aligned}
$$

There is no reason to simplify this, but multiplying top and bottom by $2\sqrt{x+y}$ gives

$$
\frac{dy}{dx} = \frac{4\,xy^2\sqrt{x+y}+1}{1-4\,x^2y\sqrt{x+y}}.
$$

2.6.14. Find dy/dx by implicit differentiation.

$$
\begin{aligned}
x\sin y+y\sin x &= 1\\
\frac{d}{dx}\left(x\sin y+y\sin x\right) &= \frac{d}{dx}1\\
\frac{d}{dx}\left(x\sin y\right)+\frac{d}{dx}\left(y\sin x\right) &= 0\\
\left[\frac{d}{dx}(x)\sin y+x\frac{d}{dx}(\sin y)\right]+\left[\frac{dy}{dx}\cdot\sin x+y\frac{d}{dx}(\sin x)\right] &= 0\\
\left[(1)\sin y+x\left(\cos y\cdot\frac{dy}{dx}\right)\right]+\left[\sin x\cdot\frac{dy}{dx}+y\cos x\right] &= 0\\
x\cos y\cdot\frac{dy}{dx}+\sin x\cdot\frac{dy}{dx} &= -\sin y-y\cos x\\
(x\cos y+\sin x)\cdot\frac{dy}{dx} &= -\sin y-y\cos x\\
\frac{dy}{dx} &= \frac{-\sin y-y\cos x}{x\cos y+\sin x}.
\end{aligned}
$$

2.6.20. Use implicit differentiation to find the equation of the tangent line to the curve $x^2+2xy-y^2+x=2$ at the point $(x,y)=(1,2)$. First we compute the slope dy/dx of the tangent line:

$$
\begin{aligned}
x^2+2xy-y^2+x &= 2\\
\frac{d}{dx}\left(x^2+2xy-y^2+x\right) &= \frac{d}{dx}2\\
\frac{d}{dx}(x^2)+2\frac{d}{dx}(xy)-\frac{d}{dx}(y^2)+\frac{d}{dx}(x) &= \frac{d}{dx}0\\
2x+2\left(1\cdot y+x\cdot\frac{dy}{dx}\right)-\left(2y\cdot\frac{dy}{dx}\right)+1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
2x + 2y + 2x\frac{dy}{dx} - 2y\frac{dy}{dx} + 1 &= 0 \\
(2x - 2y)\frac{dy}{dx} &= -1 - 2x - 2y \\
\frac{dy}{dx} &= \frac{-1 - 2x - 2y}{2x - 2y}.
\end{aligned}
$$

At the point $(x, y) = (1, 2)$ the slope becomes

$$\frac{dy}{dx} = \frac{-1 - 2(1) - 2(2)}{2(1) - 2(2)} = \frac{-7}{-2} = \frac{7}{2}.$$

Finally, the line with slope $7/2$ that passes through the point $(1, 2)$ has the equation

$$
\begin{aligned}
\frac{y-2}{x-1} &= \frac{7}{2} \\
y - 2 &= \frac{7}{2}(x-1) \\
y &= 2 + \frac{7}{2}(x-1) \\
y &= \frac{7}{2}x - \frac{7}{2} + 2.
\end{aligned}
$$

Here is a picture of the curve and the tangent line: