Math 161 — Summer 2024
Homework 1 — Drew Armstrong

Book Problems:

- Section 1.4, Exercises 8, 14, 16, 24, 26, 50, 52, 53, 56
- Section 1.6, Exercises 14, 20, 22, 28
- Section 8.1, Exercises 14, 16

1.4.8. This limit is not an indeterminate form, so we just substitute $x = 0$:

$$\lim_{x\to 0} \frac{\cos^4 x}{5 + 2x^3} = \frac{\cos^4 0}{5 + 2(0)^3} = \frac{1}{5}.$$

1.4.14. This limit has the form (nonzero number)/0 so we know it is $+\infty$ or $-\infty$, or it doesn't exist. To be sure, we will factor the numerator and denominator:

$$\begin{aligned}\lim_{x\to -1} \frac{x^2 - 4x}{x^2 - 3x - 4} &= \lim_{x\to -1} \frac{x^2 - 4x}{x^2 - 3x - 4} \\ &= \lim_{x\to -1} \frac{x\cancel{(x-4)}}{(x+1)\cancel{(x-4)}} \\ &= \lim_{x\to -1} \frac{x}{x+1}.\end{aligned}$$

As $x \to -1$ from the left, $x + 1$ is tiny and negative:

$$\lim_{x\to -1^-} \frac{x}{x+1} = \frac{-1}{\text{tiny negative}} = +\infty.$$

And as $x \to -1$ from the right, $x + 1$ is tiny and positive:

$$\lim_{x\to -1^+} \frac{x}{x+1} = \frac{-1}{\text{tiny positive}} = -\infty.$$

So the limit doesn't exist.

Remark: Problem **1.4.12.** asks for the limit of the same function as $x \to 4$, which is

$$\lim_{x\to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x\to 4} \frac{x}{x+1} = \frac{4}{5}.$$

1.4.16. This limit has the indeterminate form 0/0, so we need a trick. Factor the numerator and denominator to get

$$\begin{aligned}\lim_{x\to -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3} &= \lim_{x\to -1} \frac{(2x+1)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}} \\ &= \lim_{x\to -1} \frac{2x+1}{x-3} \\ &= \frac{2(-1)+1}{(-1)-3} \\ &= \frac{-1}{-4} \\ &= \frac{1}{4}.\end{aligned}$$

1.4.24. This limit has the indeterminate form $\infty - \infty$ so we need a trick. First we add the fractions then we can factor and cancel:

$$\begin{aligned}
\lim_{t\to 0}\left(\frac{1}{t} - \frac{1}{t^2+t}\right) &= \lim_{t\to 0}\left(\frac{t^2+t}{t(t^2+t)} - \frac{t}{t(t^2+t)}\right) \\
&= \lim_{t\to 0}\frac{t^2}{t(t^2+t)} \\
&= \lim_{t\to 0}\frac{\cancel{t^2}}{\cancel{t^2}(t+1)} \\
&= \lim_{t\to 0}\frac{1}{t+1} \\
&= \frac{1}{0+1} \\
&= 1.
\end{aligned}$$

1.4.26. This limit has the indeterminate form $0/0$ so we need a trick. We multiply the numerator and denominator by the "conjugate expression" to get

$$\begin{aligned}
\lim_{x\to -4}\frac{\sqrt{x^2+9}-5}{x+4} \quad \lim_{x\to -4}&\frac{\sqrt{x^2+9}-5}{x+4}\cdot\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5} \\
&= \lim_{x\to -4}\frac{(\sqrt{x^2+9})^2-5^2}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{x^2+9-25}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{x^2-16}{(x+4)(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{\cancel{(x+4)}(x-4)}{\cancel{(x+4)}(\sqrt{x^2+9}+5)} \\
&= \lim_{x\to -4}\frac{x-4}{\sqrt{x^2+9}+5} \\
&= \frac{(-4)-4}{\sqrt{(-4)^2+9}+5} \\
&= \frac{-8}{\sqrt{25}+5} \\
&= \frac{-8}{10} \\
&= -\frac{4}{5}.
\end{aligned}$$

1.4.50. Recall that $\sin x/x \to 1$ and $\sin(nx)/x \to n$ as $x \to 0$. Hence we have

$$\lim_{x\to 0}\frac{\sin(4x)}{\sin(6x)} = \lim_{x\to 0}\frac{\sin(4x)/x}{\sin(6x)/x} = \frac{\lim_{x\to 0}\sin(4x)/x}{\lim_{x\to 0}\sin(6x)/x} = \frac{4}{6} = \frac{2}{3}.$$

1.4.52. Recall that $(\cos x - 1)/x \to 0$ as $x \to 0$. Hence we have

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0} \frac{(\cos\theta - 1)/\theta}{\sin\theta/\theta} = \frac{\lim_{\theta\to 0}(\cos\theta - 1)/\theta}{\lim_{\theta\to 0}\sin\theta/\theta} = \frac{0}{1} = 0.$$

1.4.53. This time we factor the denominator to get an expression of the form $\sin(3x)/x$:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)}{5x^3 - 4x} &= \lim_{x\to 0} \frac{\sin(3x)}{x(5x^2-4)} \\ &= \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \frac{1}{5x^2 - 4} \\ &= \lim_{x\to 0} \frac{\sin(3x)}{x} \cdot \lim_{x\to 0} \frac{1}{5x^2-4} \\ &= 3 \cdot \frac{1}{5(0)^2 - 4} \\ &= 3 \cdot \frac{1}{-4} \\ &= -\frac{3}{4}.\end{aligned}$$

1.4.56. This limit has the indeterminate form 0/0, so we need a trick. It is difficult to change the x^2 inside the sin function so instead we will change the denominator to look like x^2:

$$\frac{\sin(x^2)}{x} = x \cdot \frac{\sin(x^2)}{x^2}.$$

If $y = x^2$ then we note that $y \to 0$ as $x \to 0$, hence

$$\lim_{x\to 0} \frac{\sin(x^2)}{x^2} = \lim_{y\to 0} \frac{\sin y}{y} = 1.$$

Finally we have

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(x^2)}{x^2} &= \lim_{x\to 0} \left(x \cdot \frac{\sin(x^2)}{x^2} \right) \\ &= \lim_{x\to 0} x \cdot \lim_{x\to 0} \frac{\sin(x^2)}{x^2} \\ &= 0 \cdot 1 \\ &= 0.\end{aligned}$$

Remark: That one was tricky. The idea with these problems is to **always look for an expression of the form** $(\sin y)/y$ where y could be something like x^2 or $3x$, etc., so that $y \to 0$ when $x \to 0$. If that doesn't work, **the only other trigonometric limit we know is** $(\cos y - 1)/y \to 0$ **when** $y \to 0$.

1.6.14. If $x \to -3$ from the left then $x + 3$ is a tiny negative number, hence

$$\lim_{x\to -3^-} \frac{x+2}{x+3} = \frac{-1}{\text{tiny negative number}} = +\infty.$$

1.6.20. This limit has the indeterminate form ∞/∞ so we need a trick. Divide the numerator and denominator by the highest power of x (in this case, x^3) to obtain

$$\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x + 1} = \lim_{x\to\infty} \frac{(1-x^2)/x^3}{(x^3 - x + 1)/x^3}$$

$$
\begin{aligned}
&= \lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{x^2}{x^3}}{\frac{x^3}{x^3} - \frac{x}{x^3} + \frac{1}{x^3}} \\
&= \lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{1}{x}}{1 - \frac{1}{x^2} + \frac{1}{x^3}} \\
&= \frac{0-0}{1-0+0} \\
&= 0.
\end{aligned}
$$

1.6.22. This is just like the previous problem, but it involves fractional exponents. First note that $t\sqrt{t} = t \cdot t^{1/2} = t^{1+1/2} = t^{3/2}$. Thus $3/2$ is the highest exponent that occurs. Divide the numerator and denominator by $t^{3/2}$ to obtain

$$
\begin{aligned}
\lim_{t\to\infty} \frac{t - t^{3/2}}{2t^{3/2} + 3t - 5} &= \lim_{t\to\infty} \frac{(t - t^{3/2})/t^{3/2}}{(2t^{3/2} + 3t - 5)/t^{3/2}} \\
&= \lim_{t\to\infty} \frac{\frac{t}{t^{3/2}} - \frac{t^{3/2}}{t^{3/2}}}{2\frac{t^{3/2}}{t^{3/2}} + 3\frac{t}{t^{3/2}} - 5\frac{1}{t^{3/2}}} \\
&= \lim_{t\to\infty} \frac{\frac{1}{t^{1/2}} - 1}{2 + 3\frac{1}{t^{1/2}} - 5\frac{1}{t^{3/2}}} \\
&= \frac{0-1}{2+3(0)-5(0)} \\
&= -\frac{1}{2}.
\end{aligned}
$$

Remark: Here we really had to remember our exponent rules:

$$
t^a \cdot t^b = t^{a+b} \quad \text{and} \quad \frac{t^a}{t^b} = t^{a-b} \quad \text{and} \quad t^{-a} = \frac{1}{t^a}.
$$

Then we used the fact that $1/t^a \to 0$ as $t \to \infty$ for any exponent $a > 0$.

1.6.28. Note that $\sin^2 x$ stays bounded as $x \to \infty$, so that

$$
\lim_{x\to\infty} \frac{\sin^2 x}{x^2} = \frac{\text{bounded number}}{\infty} = 0.
$$

This solution is perfectly acceptable. To be more precise, we can use the Squeeze Theorem. First note that $0 \le \sin^2 x \le 1$. If $x > 0$ then dividing all three terms by x preserves the direction of the inequalities:

$$
\begin{array}{ccccc}
0 & \le & \sin^2 x & \le & 1 \\
\frac{0}{x^2} & \le & \frac{\sin^2 x}{x^2} & \le & \frac{1}{x^2} \\
0 & \le & \frac{\sin^2 x}{x^2} & \le & \frac{1}{x^2}.
\end{array}
$$

Then since $1/x^2 \to 0$ as $x \to \infty$, the expression $(\sin^2 x)/x^2$ gets squeezed to zero. I don't expect you to come up with a fancy proof like this.

8.1.14. Section 8.1 is very similar to section 1.6, but with integers $n \to \infty$ instead of real numbers $x \to \infty$. There isn't much difference. In this case we use the fact that $a^n \to 0$ as $n \to \infty$ when $0 < a < 1$ to get

$$
\lim_{n\to\infty} \frac{3^{n+2}}{5^n} = \lim_{n\to\infty} \frac{3^2 \cdot 3^n}{5^n} = \lim_{n\to\infty} 3^2 \cdot \frac{3^n}{5^n} = \lim_{n\to\infty} 3^2 \cdot \left(\frac{3}{5}\right)^n = 3^2 \cdot 0 = 0.
$$

1.8.16. We can bring the limit inside the square root:

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{9n+1}}.$$

Dividing the numerator and denominator of $(n+1)/(9n+1)$ by the highest power of n (i.e., just n) we see that $(n+1)/(9n+1) = (1+1/n)/(9+1/n) \to 1/9$ as $n \to \infty$. Hence

$$\lim_{n\to\infty}\sqrt{\frac{n+1}{9n+1}} = \sqrt{\lim_{n\to\infty}\frac{n+1}{9n+1}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Remark: For any **continuous functions** $f(x)$ **and** $g(x)$ we have

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

The square root function is continuous, so we can bring limits inside it. But you would have done that anyway; there's no point making a big fuss about it.