Math 161 **Summer 2024**
Exam 4 **Drew Armstrong**

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1. Find the most general function $f(\theta)$ whose second derivative is $f''(\theta) = \sin\theta$.

Integrate once to get

$$\begin{aligned} f'(\theta) &= \int f''(\theta)\,d\theta \\ &= \int \sin\theta\,d\theta \\ &= -\cos\theta + c_1, \end{aligned}$$

then integrate again to get

$$\begin{aligned} f(\theta) &= \int f'(\theta)\,d\theta \\ &= \int (-\cos\theta + c_1)\,d\theta \\ &= -\sin\theta + c_1\theta + c_2, \end{aligned}$$

for some constants c_1 and c_2.

2. Compute the definite integral $\displaystyle\int_1^4 \frac{1}{x^2}\,dx$.

$$\begin{aligned} \int_1^4 \frac{1}{x^2}\,dx &= \int_1^4 x^{-2}\,dx \\ &= \left[\frac{x^{-2+1}}{-2+1}\right]_1^4 \\ &= \left[-\frac{1}{x}\right]_1^4 \\ &= -\frac{1}{4} - \left(-\frac{1}{1}\right) \\ &= \frac{3}{4}. \end{aligned}$$

We can interpret this as an area:[1]

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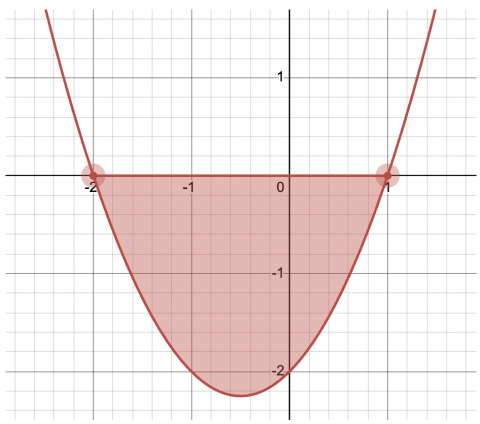
[1] https://www.desmos.com/calculator/3iexup6cp8

3. Compute the definite integral $\int_{-2}^{1} (u-1)(u+2)\,du$.

We don't yet know the backwards product rule, so I will just expand and use the backwards power rule:

$$\begin{aligned}\int_{-2}^{1} (u-1)(u+2)\,du &= \int_{-2}^{1} (u^2+u-2)\,du \\ &= \left[\frac{u^3}{3}+\frac{u^2}{2}-2u\right]_{-2}^{1} \\ &= \left[\frac{1^3}{3}+\frac{1^2}{2}-2(1)\right]-\left[\frac{(-2)^3}{3}+\frac{(-2)^2}{2}-2(-2)\right] \\ &= \left[\frac{1}{3}+\frac{1}{2}-2\right]-\left[\frac{-8}{3}+2+4\right] \\ &= -\frac{9}{2}.\end{aligned}$$

The result is negative because the graph is below the u-axis when $- < 2 < u < 1$:[2]



4. Use the Fundamental Theorem of Calculus to find the derivative $f'(x)$ of the function

$$f(x) = \int_0^{\sqrt{x}} \sin t\,dt.$$

The FTC says that

$$g(u) = \int_0^{u} \sin t\,dt \quad \Longrightarrow \quad g'(u) = \sin u.$$

But we are interested in the function $f(x) = g(\sqrt{x})$ so we use the chain rule:

$$f'(x) = [g(\sqrt{x})]' = g'(\sqrt{x}) \cdot (\sqrt{x})' = \sin(\sqrt{x}) \cdot \left(\frac{1}{2\sqrt{x}}\right).$$

[2] https://www.desmos.com/calculator/udvixvbqu7

5. Use substitution to find the antiderivative $\int x \cdot (x^2+1)^{19}\,dx$.

Let $m = x^2 + 1$ so that $dm = 2x\,dx$ and $dx = dm/(2x)$. Then we have

$$\begin{aligned}
\int x \cdot (x^2+1)^{19}\,dx &= \int x \cdot m^{19}\,dx \\
&= \int \cancel{x} \cdot m^{19} \cdot \frac{dm}{2\cancel{x}} \\
&= \frac{1}{2}\int m^{19}\,dm \\
&= \frac{1}{2} \cdot \frac{m^{20}}{20} + c \\
&= \frac{1}{40}(x^2+1)^{20} + c,
\end{aligned}$$

for some constant c.