Math 161 **Summer 2024**
Exam 3 **Drew Armstrong**

---

1. The volume V of a cube is increasing at a constant rate of 2 cm^3/sec. At what rate is the side length increasing when $V = 8$ cm^3 ?

Note that $V = \ell^3$, where ℓ is the side length of the cube. Taking time derivatives on both sides of the equation $V = \ell^3$ gives

$$V' = 3\ell^2\ell'.$$

We are given that $V' = 2$ and $V = 8$, which implies that $\ell = \sqrt[3]{8} = 2$. Hence

$$\ell' = \frac{V'}{3\ell^2} = \frac{2}{3(2)^2} = \frac{1}{6} \quad \text{cm}^3 \text{ / sec.}$$

2. Use linear approximation to estimate the value of $\dfrac{1}{1.9}$.

Let $f(x) = 1/x$ so that $f'(x) = -1/x^2$. Then for any x close to 2 we have

$$\begin{aligned} f(x) &\approx f(2) + f'(2)(x-2), \\ \frac{1}{x} &\approx \frac{1}{2} - \frac{1}{4}(x-2). \end{aligned}$$

Since 1.9 is close to 2 we have

$$\frac{1}{1.9} \approx \frac{1}{2} - \frac{1}{4}(1.9-2) = \frac{1}{2} - \frac{1}{4}(-0.1) = 0.5 + 0.025 = 0.525.$$

Remark: The exact value is $1/1.9 = 0.52632$.

3. The area A of a hemispherical dome is $C^2/(2\pi)$, where C is the circumference. Suppose we measure C to be 10 ± 0.1 meters. Estimate the uncertainty in the calculated value of A.

Note that $dA/dC = 2C/(2\pi)$, and hence

$$dA = \frac{2C}{2\pi} \cdot dC = \frac{C}{\pi} \cdot dC.$$

We are given that $C = 10$ and $dC = 0.1$, hence

$$dA = \frac{10}{\pi} \cdot (0.1) = \frac{1}{\pi}.$$

Remark: The relative error is more meaningful. We have $A = (10)^2/(2\pi)$ and hence

$$\frac{dA}{A} = \frac{1/\pi}{100/2\pi} = \frac{2}{100} = 2\%.$$

4. Suppose that $xy = 1$. In this case, find the **minimum** possible value of $x + 4y$. (Assume that x and y are positive.)

We want to minimize the function $f(x, y) = x + 4y$. To do this we first use the equation $xy = 1$ to eliminate y:

$$f(x) = x + 4\left(\frac{1}{x}\right) = x + \frac{4}{x}.$$

Then we set the first derivative $f'(x)$ equal to zero:

$$\begin{aligned} f'(x) &= 0 \\ 1 - \frac{4}{x^2} &= 0 \\ x^2 - 4 &= 0 \\ x &= \pm 2. \end{aligned}$$

But we assumed that x is positive, so we must have $x = 2$ and $y = 1/2$, and the minimum value of $x + 4y$ is $2 + 4/2 = 4$.

5. Find all the inflection points on the graph of $f(x) = x^4 - 2x^3$. (Give the x and y coordinates of the inflection points.)

Inflection points occur where the second derivative is zero. First we compute

$$\begin{aligned} f(x) &= x^4 - 2x^3, \\ f'(x) &= 4x^3 - 6x^2, \\ f''(x) &= 12x^2 - 12x = 12x(x-1). \end{aligned}$$

We note that $f''(x) = 0$ implies $x = 0$ or $x = 1$. Hence there are two inflection points:

$$(0, f(0)) = (0, 0) \quad \text{and} \quad (1, f(1)) = (1, -1).$$

Here is a picture: