Math 161 **Summer 2023**
Final Exam **Drew Armstrong**

1. $\lim_{x\to 2} \dfrac{x^2+1}{x^2-1}$.

This limit is not an indeterminate form. Just substitute $x=2$:

$$\lim_{x\to 2} \frac{x^2+1}{x^2-1} = \frac{2^2+1}{2^2-1} = \frac{5}{3}.$$

2. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

This limit has the indeterminate form 0/0. Factor the denominator and then cancel:

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{\cancel{x-2}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$$

3. $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

This limit has the indeterminate form 0/0. Multiply and divide by the conjugate expression:

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} &= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\
&= \lim_{h\to 0} \frac{(x+h)-(x)}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} \\
&= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}$$

Remark: We just computed the derivative of $\sqrt{x}$.

4. $\lim_{\theta\to 0} \dfrac{\sin(2\theta)}{\sin\theta}$

Here we use the fact that $\sin(n\theta)/\theta \to n$ as $\theta \to 0$:

$$\lim_{\theta\to 0} \frac{\sin(2\theta)}{\sin\theta} = \lim_{\theta\to 0} \frac{\sin(2\theta)/\theta}{(\sin\theta)/\theta} = \frac{2}{1} = 2.$$

5. $\lim_{x\to 2^+} \dfrac{x-1}{x-2}$

This limit has the form (nonzero)/0, so it is $\pm\infty$. To determine which we must investigate the sign of the numerator and denominator:

$$\lim_{x\to 2^+} \frac{x-1}{x-2} = \frac{1}{\text{tiny positive number}} = +\infty.$$

6. $\lim_{n\to\infty} \dfrac{3n^3}{n(n^2-1)}$

Expand and then divide numerator and denominator by the highest power of n:

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^3}{n(n^2-1)} &= \lim_{n\to\infty} \frac{3n^3}{n^3-n} \\
&= \lim_{n\to\infty} \frac{(3n^3)/n^3}{(n^3-n)/n^3} \\
&= \lim_{n\to\infty} \frac{3}{1-1/n^2} \\
&= \frac{3}{1-0} \\
&= 3.
\end{aligned}$$

7. Compute the derivative of $f(x) = (x+1)(x+2)$.

Expand and use the power rule:

$$\begin{aligned}
f(x) &= x^2 + 3x + 2 \\
f'(x) &= 2x + 3.
\end{aligned}$$

Or use the product rule:

$$\begin{aligned}
f(x) &= (x+1)(x+2) \\
f'(x) &= (x+1)(x+2)' + (x+1)'(x+2) \\
&= (x+1)(1) + (1)(x+2) \\
&= 2x + 3.
\end{aligned}$$

8. Compute the derivative of $f(x) = \sqrt{x^3+1}$.

Use the power rule and the chain rule:

$$\begin{aligned}
f(x) &= (x^3+1)^{1/2} \\
f'(x) &= \frac{1}{2}(x^3+1)^{1/2-1} \cdot (x^3+1)' \\
&= \frac{1}{2}(x^3+1)^{-1/2} \cdot (3x^2) \\
&= \frac{3x^2}{2\sqrt{x^3+1}}.
\end{aligned}$$

9. Compute the derivative of $f(x) = \dfrac{\sin x}{\cos x}$.

Use the quotient rule:

$$\begin{aligned}
f(x) &= \frac{\sin x}{\cos x} \\
f'(x) &= \frac{(\cos x)(\sin x)' - (\sin x)(\cos x)'}{(\cos x)^2}
\end{aligned}$$

$$= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$
$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2}$$
$$= \frac{1}{(\cos x)^2}.$$

10. Compute the derivative of $f(x) = (x+1)^5(x+2)^7$.

There is no way I'm going to expand this, so let's use the product rule and chain rule:

$$\begin{aligned} f(x) &= (x+1)^5(x+2)^7 \\ f'(x) &= (x+1)^5((x+2)^7)' + ((x+1)^5)'(x+2)^7 \\ &= (x+1)^5(7(x+2)^6(1)) + (5(x+1)^4(1))(x+2)^7. \end{aligned}$$

I guess we could simplify this:

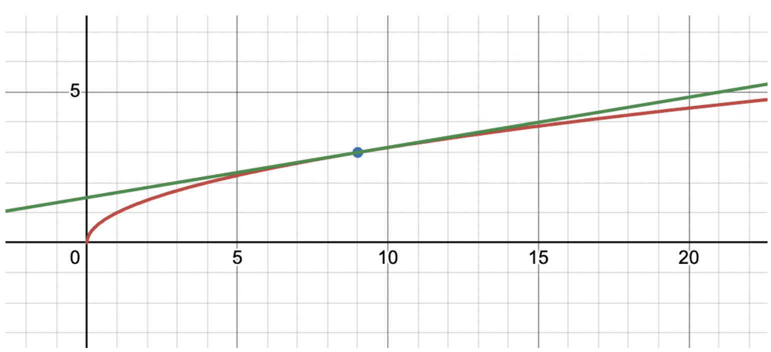
$$\begin{aligned} f'(x) &= (x+1)^4(x+2)^6(7(x+1) + 5(x+2)) \\ &= (x+1)^4(x+2)^6(12x+17). \end{aligned}$$

11. Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point $(x, y) = (9, 3)$.

The slope of the tangent line at the point $(x, \sqrt{x})$ is $dy/dx = \frac{1}{2\sqrt{x}}$. At the point $(9, 3)$ this is $dy/dx = \frac{1}{2\sqrt{9}} = 1/6$. Hence the equation of the tangent line is

$$\begin{aligned} \frac{y-3}{x-9} &= \frac{1}{6} \\ y - 3 &= \frac{1}{6}(x-9) \\ y &= 3 + \frac{1}{6}x - \frac{3}{2} \\ y &= \frac{1}{6}x + \frac{3}{2}. \end{aligned}$$

Here is a picture:



12. Use linear approximation to estimate the value of $\sqrt[3]{8.1}$.

We know that $\sqrt[3]{8} = 2$. This suggests we should approximate the function $f(x) = \sqrt[3]{x}$ near $a = 8$. Note that $f'(x) = \frac{1}{3}x^{-2/3}$. The linear approximation formula is

$$\begin{aligned} f(x) &\approx f(8) + f'(8)(x-8) \\ \sqrt[3]{x} &\approx \sqrt[3]{8} + \frac{1}{3}8^{-2/3}(x-8) \\ \sqrt[3]{x} &\approx 2 + \frac{1}{12}(x-8). \end{aligned}$$

This approximation is good for $x \approx 8$. Since $8.1 \approx 8$ we have

$$\sqrt[3]{8.1} \approx 2 + \frac{1}{12}(0.1).$$

13. Consider a circle with radius r. Suppose the area of the circle is increasing at a constant rate of 1 cm^2 per second. At what rate is r increasing when $r = 2$ cm?

Let A be the area of the circle so that $A = \pi r^2$. Taking time derivatives gives

$$\begin{aligned} (A)' &= (\pi r^2)' \\ A' &= \pi(r^2)' \\ A' &= \pi(2rr'). \end{aligned}$$

If $A' = 1$ and $r = 2$ then we have

$$\begin{aligned} 1 &= \pi(2 \cdot 2 \cdot r') \\ 1 &= 4\pi r' \\ r' &= \frac{1}{4\pi}. \end{aligned}$$

14. Consider a right-angled triangle with side lengths a, b, c, where c is the hypotenuse. Suppose that a and b are measured to be 5 cm and 10 cm, each with a maximum error of 0.1 cm. In this case, estimate the maximum error in the calculated value of c.

The Pythagorean theorem says that $c^2 = a^2 + b^2$. Taking differentials gives

$$\begin{aligned} d(c^2) &= d(a^2 + b^2) \\ 2c\,dc &= 2a\,da + 2b\,db. \end{aligned}$$

Then substituting $a = 5$, $b = 10$ and $da = db = 0.1$ gives

$$\begin{aligned} 2c\,dc &= 2(5)(0.1) + 2(10)(0.1) \\ dc &= \frac{(5)(0.1) + (10)(0.1)}{c} \\ dc &= \frac{(5)(0.1) + (10)(0.1)}{\sqrt{5^2 + 10^2}}. \end{aligned}$$

15. Find the maximum value of xy subject to the constraint $2x + 3y = 1$.

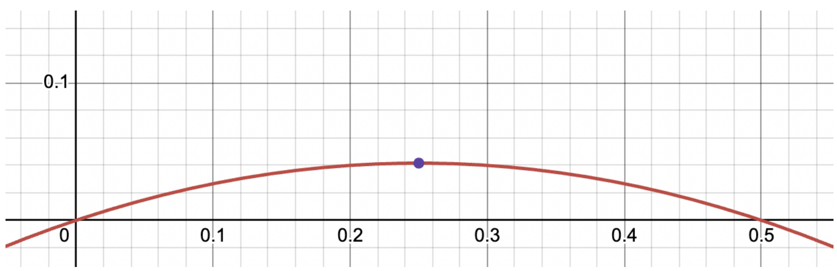
We want to maximize the function $f(x, y) = xy$. First we use the constraint $2x + 3y = 1$ to eliminate y. We have $y = (1 - 2x)/3$ and hence

$$f(x) = x \cdot \frac{1 - 2x}{3}.$$

Set the derivative $f'(x)$ equal to zero and solve for x:

$$\begin{aligned} f'(x) &= 0 \\ (x)' \cdot \frac{1-2x}{3} + x \cdot \left(\frac{1-2x}{3}\right)' &= 0 \\ \frac{1-2x}{3} + x \cdot \left(-\frac{2}{3}\right) &= 0 \\ \frac{1-4x}{3} &= 0 \\ 1-4x &= 0 \\ x &= \frac{1}{4}. \end{aligned}$$

It follows that $y = (1-2(1/4))/3 = 1/6$ and the maximum value of xy is $(1/4)(1/6) = 1/24$. Here is the graph of $f(x)$:

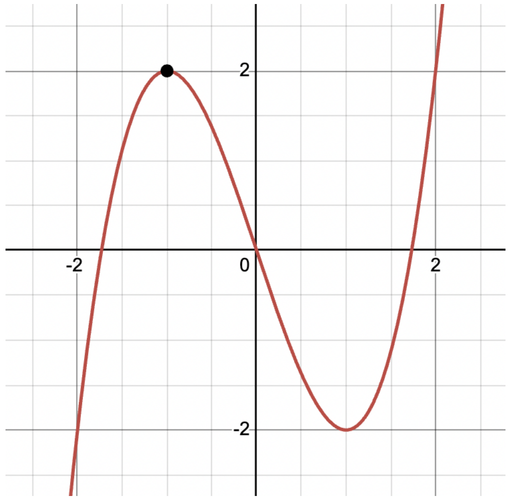


16. For which value of x does $f(x) = x(x^2-3)$ attain a local maximum?

Set the derivative $f'(x)$ equal to zero:

$$\begin{aligned} f'(x) &= 0 \\ (x^3-3x)' &= 0 \\ 3x^2-3 &= 0 \\ 3(x^2-1) &= 0 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0. \end{aligned}$$

Hence $f'(x) = 0$ implies $x = 1$ or $x = -1$. In between we have $f'(x) < 0$ when $-1 < x < 1$ and $f'(x) > 0$ otherwise. Around $x = -1$, $f'(x)$ switches from positive to negative, hence this is a local maximum. Alternatively, the second derivative $f''(x) = 2x$ is negative at $x = -1$, which confirms that this is a local max. Picture:



17. Find the most general function $f(t)$ whose second derivative is $f''(t) = 5$.

Integrating once gives

$$\begin{aligned} f'(t) &= \int f''(t)\,dt \\ &= \int 5\,dt \\ &= 5t + c_1, \end{aligned}$$

then integrating twice gives

$$\begin{aligned} f(t) &= \int f'(t)\,dt \\ &= \int (5t + c_1)\,dt \\ &= 5\frac{1}{2}t^2 + c_1 t + c_2, \end{aligned}$$

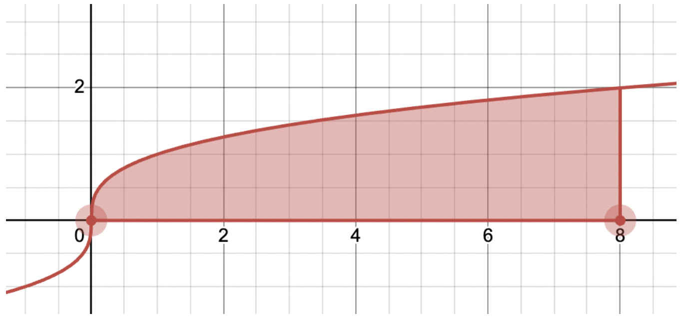
for some constants c_1 and c_2.

18. Compute the definite integral $\int_0^8 \sqrt[3]{x}\,dx$.

First we note that $F(x) = \frac{x^{1/3+1}}{1/3+1} = \frac{3}{4}x^{4/3}$ is an antiderivative of $\sqrt[3]{x} = x^{1/3}$. The the Fundamental Theorem of Calculus gives

$$\int_0^8 \sqrt[3]{x}\,dx = F(8) - F(0) = \frac{3}{4} \cdot 8^{4/3} - \frac{3}{4} \cdot 0^{4/3} = \frac{3}{4} \cdot 16 - 0 = 12.$$

We can view this as an area:

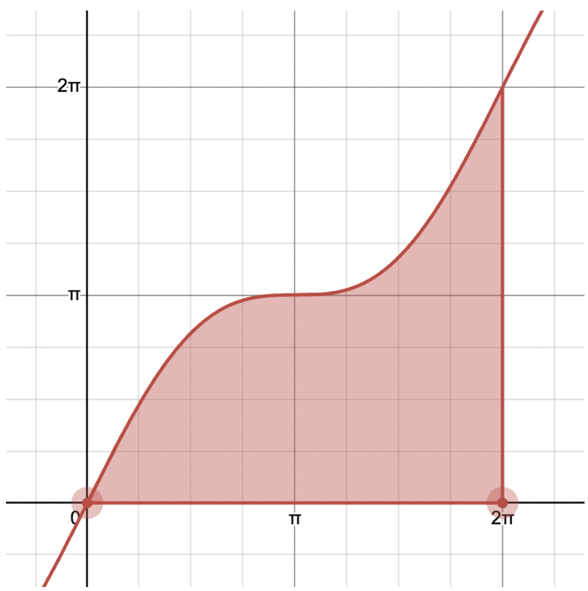


19. Compute the definite integral $\int_0^{2\pi} (\theta + \sin\theta)\, d\theta$.

Just do it:

$$\begin{aligned}\int_0^{2\pi} (\theta + \sin\theta)\, d\theta &= \left(\frac{1}{2}\theta^2 - \cos\theta\right)_0^{2\pi} \\ &= \left(\frac{1}{2}(2\pi)^2 - \cos 2\pi\right) - (0 - \cos 0) \\ &= \left(2\pi^2 - 1\right) - (0 - 1) \\ &= 2\pi^2.\end{aligned}$$

We can view this as an area:[1]



20. Use the Fundamental Theorem of Calculus to find the derivative $f'(x)$ of the function

$$f(x) = \int_7^{x^2} \sin t\, dt.$$

[1] `https://www.desmos.com/calculator/fqhfkzzf2c`

The FTC says that

$$g(u) = \int_7^u \sin t\,dt \quad \Longrightarrow \quad g'(u) = \sin u.$$

Since $f(x) = g(x^2)$, the chain rule says that

$$f'(x) = [g(x^2)]' = g'(x^2)(x^2)' = \sin(x^2) \cdot 2x.$$

21. Use substitution to find the antiderivative $\int x^2\sqrt{x^3+1}\,dx$.

Let $u = x^3 + 1$ so that $du = 3x^2\,dx$ and $dx = du/(3x^2)$. Then we have

$$\begin{aligned}
\int x^2\sqrt{x^3+1}\,dx &= \int x^2\sqrt{u} \cdot \frac{du}{3x^2} \\
&= \frac{1}{3} \cdot \int \sqrt{u}\,du \\
&= \frac{1}{3} \cdot \int u^{1/2}\,du \\
&= \frac{1}{3} \cdot \frac{u^{1/2+1}}{1/2+1} + c \\
&= \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} + c \\
&= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} + c \\
&= \frac{2}{9}(x^3+1)^{3/2} + c.
\end{aligned}$$